On Jankov-de Jongh formulas

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The Heyting day
dedicated to Dick de Jongh and Anne Troelstra

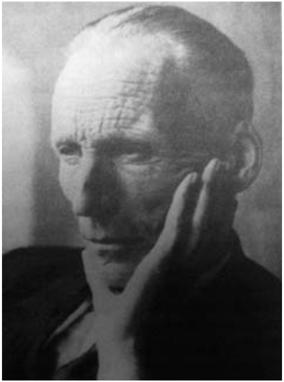
Outline

- Intermediate logics
- Varieties of Heyting algebras
- Jankov-de Jongh formulas
- Their applications
- Further generalizations

Intermediate logics

Constructive reasoning

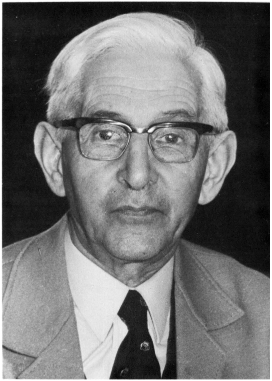
On the grounds that the only accepted reasoning should be constructive, L. E. J. Brouwer rejected classical reasoning.



Luitzen Egbertus Jan Brouwer (1881 - 1966)

Intuitionistic logic

In 1930's Brouwer's ideas led his student Heyting to introduce intuitionistic logic which formalizes constructive reasoning.



Arend Heyting (1898 - 1980)

Intuitionistic logic

Roughly speaking, the axiomatization of intuitionistic logic is obtained by dropping the law of excluded middle from the axiomatization of classical logic.

CPC = classical propositional calculus
IPC = intuitionistic propositional calculus.

The law of excluded middle is not derivable in intuitionistic logic. So **IPC** $\subsetneq$ **CPC**.

In fact,

$$\mathbf{CPC} = \mathbf{IPC} + (p \vee \neg p).$$

There are many logics in between **IPC** and **CPC**

Superintuitionistic logics

A superintuitionistic logic is a set of formulas containing **IPC** and closed under the rules of substitution and Modus Ponens.

Superintuitionistic logics contained in **CPC** are often called intermediate logics because they are situated between **IPC** and **CPC**.

Intermediate logics are exactly the consistent superintuitionistic logics.

Since we are interested in consistent logics, we will mostly concentrate on intermediate logics.

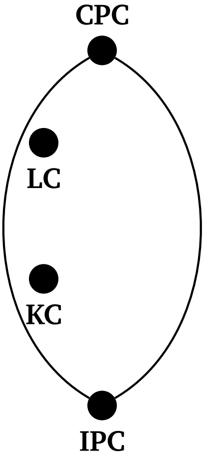
Intermediate logics

$\mathbf{LC} = \mathbf{IPC} + (p \to q) \vee (q \to p)$

Gödel-Dummett calculus

$\mathbf{KC} = \mathbf{IPC} + (\neg p \vee \neg\neg p)$

weak law of excluded middle



Varieties of Heyting algebras

Heyting algebras

A Heyting algebra is a bounded distributive lattice $(A, \wedge, \vee, 0, 1)$ equipped with a binary operation $\to$, which is a right adjoint of $\wedge$. This means that for each $a, b, x \in A$ we have

$$a \wedge x \leq b \text{ iff } x \leq a \to b.$$

Equational theories of Heyting algebras

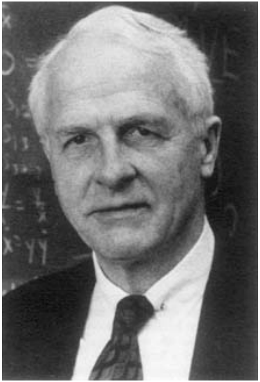
Each formula φ in the language of **IPC** corresponds to an equation $\varphi \approx 1$ in the theory of Heyting algebras.

Conversely, each equation $\varphi \approx \psi$ can be rewritten as $\varphi \leftrightarrow \psi \approx 1$, which corresponds to the formula $\varphi \leftrightarrow \psi$.

This yields a one-to-one correspondence between superintuitionistic logics and equational theories of Heyting algebras.

Varieties of Heyting algebras

By the celebrated Birkhoff theorem, equational theories correspond to varieties; that is, classes of algebras closed under homomorphic images, subalgebras, and products.



Garrett Birkhoff (1911 - 1996)

Varieties of Heyting algebras

Thus, superintuitionistic logics correspond to varieties of Heyting algebras, while intermediate logics to non-trivial varieties of Heyting algebras.

Heyt = the variety of all Heyting algebras.

Bool = the variety of all Boolean algebras.

$\Lambda(\mathbf{IPC})$ = the lattice of superintuitionistic logics.

$\Lambda(\mathbf{Heyt})$ = the lattice of varieties of Heyting algebras.

Theorem. $\Lambda(\mathbf{IPC})$ is dually isomorphic to $\Lambda(\mathbf{Heyt})$.

Consequently, we can investigate superintuitionistic logics by means of their corresponding varieties of Heyting algebras.

Properties of intermediate logics

Axiomatization, the finite model property (fmp) and decidability are some of the most studied properties of non-classical logics.

(Harrop, 1957) If a logic is finitely axiomatizable and has the fmp, then it is decidable.

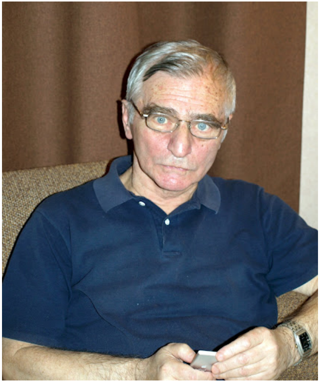
In the 1960's the research on axiomatization and finite model property was mostly concerned with particular non-classical logics.

Since the 1970's general methods started to develop for classes of non-classical logics.

One of the important aziomatization methods developed at that time was the method of Jankov-de Jongh formulas.

Aims of Jankov

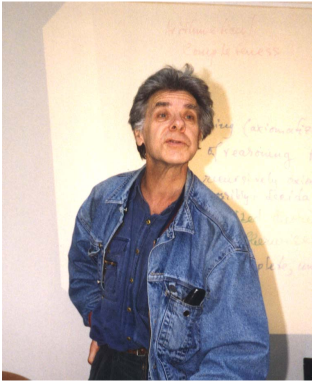
The aim of Jankov was to show that there exist continuum many intermediate logics and to construct intermediate logics without the finite model property.



Dimitri Jankov

Aims of de Jongh

The aim of de Jongh was to characterize intuitonistic logic (among all the intermediate logics) via the Kleene slash.



Dick de Jongh

Aims of de Jongh

Aims of de Jongh

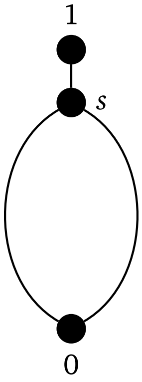
The PhD defence of Wim Blok 1976

Jankov formulas

Subdirectly irreducible Heyting algebras

By another theorem of Birkhoff, every variety of algebras is generated by its subdirectly irreducible members.

Theorem (Jankov, 1963). A Heyting algebra is subdirectly irreducible (s.i. for short) iff it has the second largest element.



Jankov formulas

Let A be a finite subdirectly irreducible Heyting algebra, s the second largest element of A.

For each $a \in A$ we introduce a new variable p_a and define the Jankov formula $\chi(A)$ as the $(\wedge, \vee, \to, 0, 1)$-description of this algebra.

$$\begin{array}{rcl}
\chi(A) & = & [\bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\
 & & \bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\
 & & \bigwedge\{p_{a\to b} \leftrightarrow p_a \to p_b : a, b \in A\} \wedge \\
 & & \bigwedge\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}] \to p_s
\end{array}$$

If we interpret p_a as a, then the Jankov formula of A is equal in A to s, i.e., it is pre-true in A .

Axiomatization of varieties of Heyting algebras

Theorem (Jankov, 1963). Let A be a finite s.i. Heyting algebra, and B be a Heyting algebra. Then

$B \not\models \chi(A)$ iff there is a homomorphic image C of B and a Heyting embedding $h : A \rightarrowtail C$.

Splittings

Jankov formulas are used to axiomatize many varieties of Heyting algebras.

For example, they axiomatize all splitting varieties of Heyting algebras.

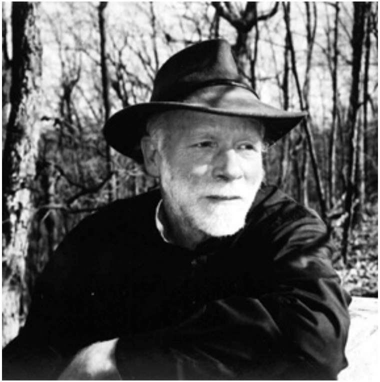
Splittings started to play an important role in lattice theory in the 1940s.

A pair (a, b) splits a lattice L if $a \nleq b$ and for each $c \in L$:

$$a \leq c \;\text{ or }\; c \leq b$$

Splittings

R. McKenzie in the 1970's revisited splittings when he started an extensive study of lattices of varieties.



Ralph McKenzie

Splittings

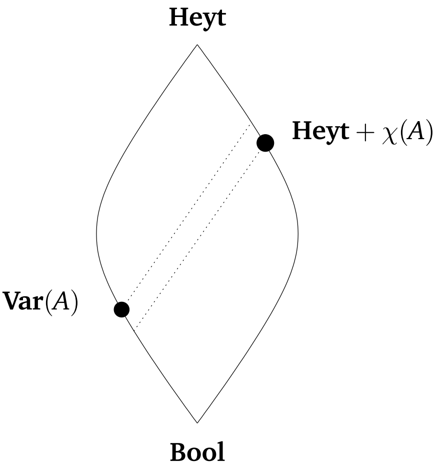


Figure: Splitting of the lattice of varieties of Heyting algebras

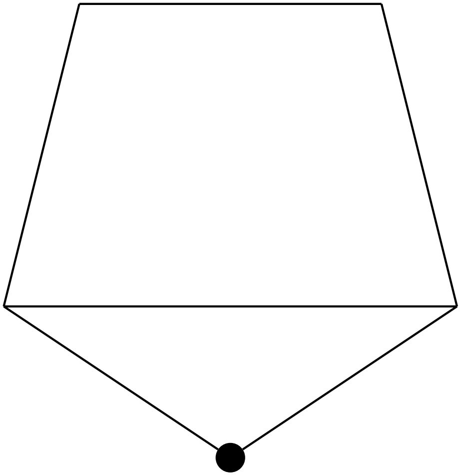
Splittings

Theorem. For each subdirectly irreducible Heyting algebra A the pair $(\mathbf{Var}(A), \mathbf{Heyt} + \chi(A))$ splits the lattice of varieties of Heyting algebras.

de Jongh formulas

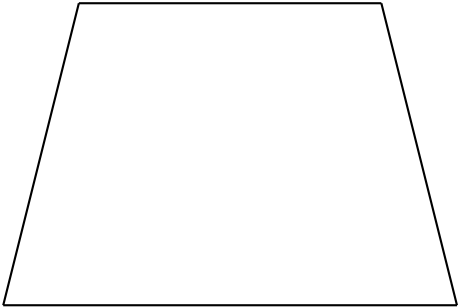
n-Henkin model of **IPC**

$H(n)$



n-universal model of **IPC**

$U(n)$

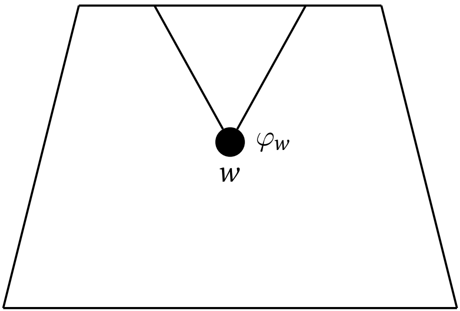


n-universal model $U(n)$ consists of the elements of $H(n)$ that have finitely many successors.

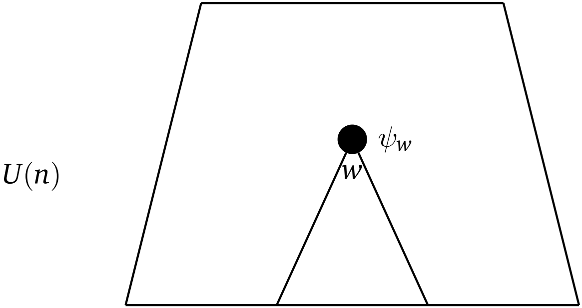
$U(n)$ is dense in $H(n)$.

n-universal model of **IPC**

$U(n)$



n-universal model of **IPC**



de Jongh formulas

De Jongh formulas φ_w and ψ_w define point-generated up-sets of $U(n)$.

In particular, $\uparrow w = V(\varphi_w)$ and $U(n) \setminus \downarrow w = V(\psi_w)$

$$\psi_w = \varphi_w \to \bigvee_{u \in S} \varphi_u$$

where S is the set of all immediate successors of w.

De Jongh formulas

Theorem (de Jongh, 1968) For any finite rooted frame $\mathfrak{F}$ there exists a formula $\chi(\mathfrak{F})$ such that for any frame $\mathfrak{G}$ we have

$\mathfrak{G} \not\models \chi(\mathfrak{F})$ iff $\mathfrak{F}$ is a bounded morphic image of a generated subframe of $\mathfrak{G}$.

Disjunction property for intermediate logics

An intermediate logic L has the disjunction property if
$L \vdash \varphi \vee \psi$ implies $L \vdash \varphi$ or $L \vdash \psi$.

Theorem. (Lukasiewicz, 1952) **IPC** has the disjunction property.

Disjunction property for intermediate logics

Conjecture. (Lukaisewicz, 1952) An intermediate logic has the disjunction property iff $L = \mathbf{IPC}$.

Jan Lukasiewicz (1978-1956)

The disjunction property

The Kreisel-Putnam Logic

$$\mathbf{KP} = \mathbf{IPC} + (\neg p \to q \vee r) \to (\neg p \to q) \vee (\neg p \to r)$$

is a proper intermediate logic that has the disjunction property.

Gabbay-de Jongh Logics provide an infinite family of intermediate logics with the disjunction property.

Wronski proved that in fact there are continuum many intermediate logics with the disjunction property.

The disjunction property

Theorem (de Jongh, 1968)
Let L be an intermediate logic. Then $L = \mathbf{IPC}$ iff for every formula φ, $\varphi|_L\varphi$ iff φ has the L-disjunction property.

The connection of Jankov and de Jongh formulas

Heyting algebra of up-sets

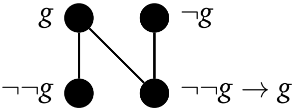
Up-sets of any poset (intuitionistic Kripke frame) $(X, \leq)$ form a Heyting algebra where for up-sets $U, V \subseteq X$:

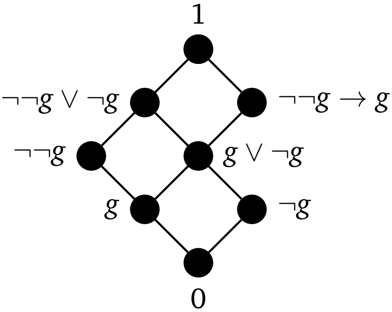
$$U \to V = X - {\downarrow}(U - V), \quad \neg U = X - {\downarrow}U$$

Here U is an up-set if $x \in U$ and $x \leq y$ imply $y \in U$ and

${\downarrow}U = \{x \in X : \exists y \in U \text{ with } x \leq y\}$.

Heyting algebra of up-sets





Heyting algebras of up-sets

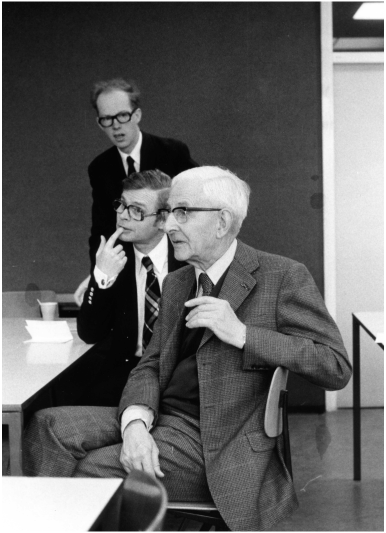
De Jongh and Troelstra gave a characterization of Heyting algebras arising from Kripke frames.

Theorem (de Jongh and Troelstra, 1966). A Heyting algebra A is isomorphic to $\mathrm{Up}(X)$ for some poset X iff A is complete and every element of A is a join of completely join-prime elements.

Corollary. Every finite Heyting algebra is isomorphic to $\mathrm{Up}(X)$ for a finite poset X.

Heyting algebras of up-sets

Anne Troelstra



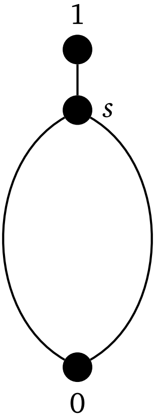
Arend Heyting and Anne Troelstra

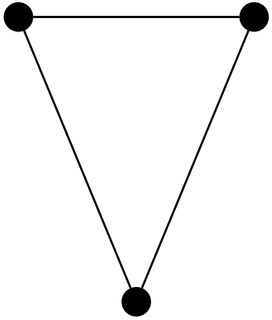
Representation of Heyting algebras

Theorem (Esakia, 1974). Every Heyting algebra is isomorphic to the Heyting algebra of clopen up-sets of some topological Kripke frame.

Leo Esakia (1934 - 2010)

Posets dual to s.i. Heyting algebras





A finite Heyting algebra A is s.i. iff the dual poset of A has a least element, the root.

Duality dictionary in the finite case

Heyting algebras	posets
s.i. Heyting algebras	rooted posets
homomorphic images	up-sets
subalgebras	bounded morphic images
Jankov formulas	de Jongh formulas

Jankov formulas and the cardinality of the lattice of intermediate logics

Continuum of intermediate logics

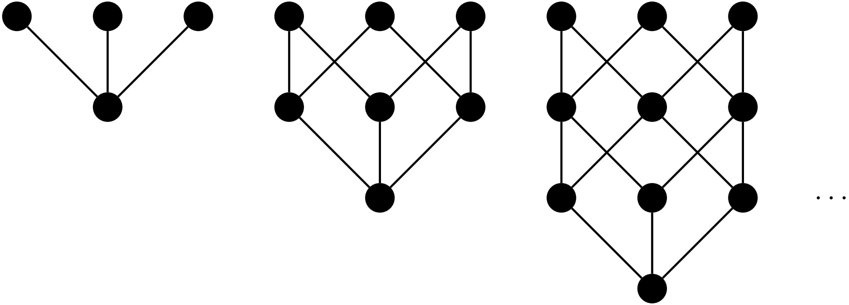
Let A and B be s.i. Heyting algebras. We write $A \leq B$ if $A \in \mathbf{SH}(B)$.

Theorem. If Δ is an $\leq$-antichain of finite s.i. algebras, then for each $I, J \subseteq \Delta$ with $I \neq J$, we have

$$\mathbf{IPC} + \{\chi(A) : A \in I\} \neq \mathbf{IPC} + \{\chi(A) : A \in J\}.$$

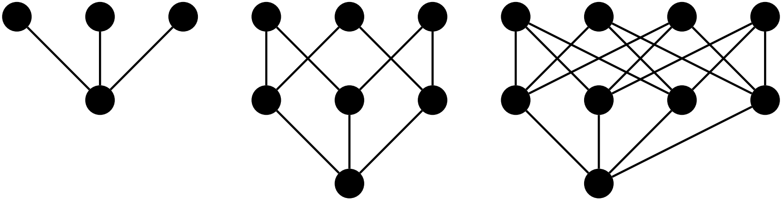
How can we construct an $\leq$-antichain of finite s.i. algebras?

Antichains



Lemma. Δ_1 is an $\leq$-antichain.

Antichains



. . .

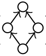
Lemma. Δ_2 is an $\leq$-antichain.

Continuum of intermediate logics

Corollary.

1. There are continuum many intermediate logics.
2. In fact, there are continuum many intermediate logics of depth 3.
3. And there are continuum many intermediate logics of width 3.

Logics axiomatized by Jankov-de Jongh formulas

- **CPC** $=$ **IPC** $+ \chi($$)$,
- **KC** $=$ **IPC** $+ \chi($$)$,
- **LC** $=$ **IPC** $+ \chi($$) + \chi($$)$.

Varieties axiomatized by Jankov formulas

Is every variety of Heyting algebras axiomatized by Jankov formulas?

A variety **V** is locally finite if every finitely generated **V**-algebra is finite.

Theorem Every locally finite variety of Heyting algebras is axiomatized by Jankov formulas.

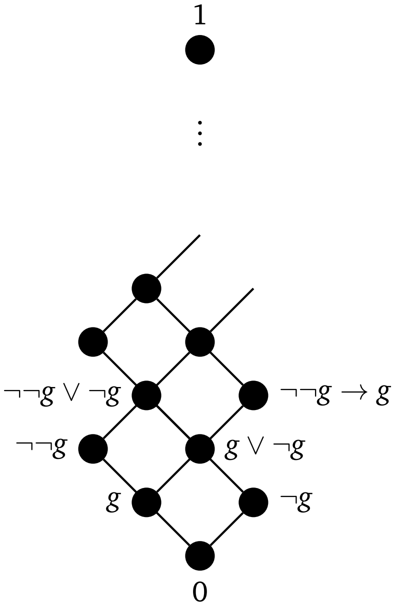
Corollary. Varieties of Heyting algebras of finite depth are locally finite and hence axiomatized by Jankov formulas.

Finitely generated algebras

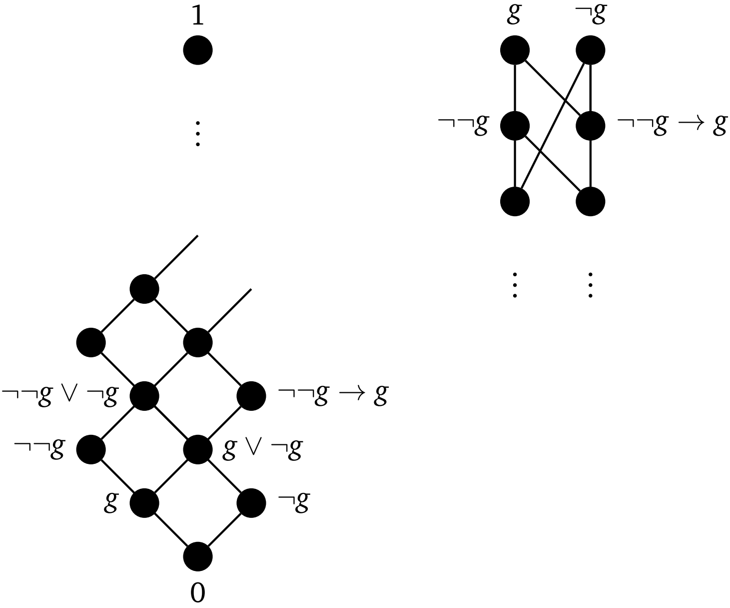
However, there are continuum many non-locally finite varieties of Heyting algebras.

Theorem (Rieger, 1949, Nishimura, 1960). The 1-generated free Heyting algebra, also called the Rieger-Nishimura lattice, is infinite.

The Rieger-Nishimura Lattice



1-generated free Heyting algebra



Axiomatization of varieties of Heyting algebras

There exist intermediate logics that are not axiomatized by Jankov-de Jongh formulas.

Problem: Can we generalize the Jankov-de Jongh method to all intermediate logics?

Canonical formulas

Axiomatization of intermediate logics

The affirmative answer was given by Michael Zakharyaschev via canonical formulas.

Michael Zakharyaschev

Duality dictionary in the finite case

Heyting algebras	posets
s.i. Heyting algebras	rooted posets
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subalgebras	bounded morphic images
Jankov formulas	de Jongh formulas
?	canonical formulas

Canonical formulas and the fmp

We will give an algebraic account of this method.

It turns out that the method of canonical formulas is directly related to the finite model property of **IPC**.

The finite model property of **IPC** for Heyting algebras is established via locally finite reducts of Heyting algebras.

Locally finite reducts

Although Heyting algebras are not locally finite, they have locally finite reducts.

Heyting algebras $(A, \wedge, \vee, \rightarrow, 0, 1)$.

$\vee$-free reducts $(A, \wedge, \rightarrow, 0, 1)$: implicative semilattices.

$\rightarrow$-free reducts $(A, \wedge, \vee, 0, 1)$: distributive lattices.

Theorem.

- (Diego, 1966). The variety of implicative semilattices is locally finite.
- (Folklore). The variety of distributive lattices is locally finite.

Connection with filtrations

There are two standard methods for proving the finite model property for modal and intermediate logics: standard filtration and selective filtration.

Taking the $(\wedge, \to, 0)$-reduct corresponds to selective filtration.

Taking the $(\wedge, \vee, 0, 1)$-reduct corresponds to standard filtration.

$(\wedge, \to)$-canonical formulas

We will use these reducts to derive desired axiomatizations of varieties of Heyting algebras.

First we will need to extend the theory of Jankov formulas.

Jankov formulas describe the full Heyting signature. We will now look at the $\vee$-free reducts.

The homomorphisms will now preserve only $\wedge$, 0 and $\to$. In general they do not preserve $\vee$. But they may preserve some joins.

This can be encoded in the following formula.

$(\wedge, \to)$-canonical formulas

Let A be a finite subdirectly irreducible Heyting algebra, s the second largest element of A, and D a subset of A^2.

For each $a \in A$ we introduce a new variable p_a and define the $(\wedge, \to)$-canonical formula $\alpha(A, D)$ associated with A and D as

$$\begin{array}{rcl} \alpha(A,D) & = & [\bigwedge\{p_{a\wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\}\wedge \\ & & \bigwedge\{p_{a\to b} \leftrightarrow p_a \to p_b : a, b \in A\}\wedge \\ & & \bigwedge\{p_{\neg a} \leftrightarrow \neg p_a : a \in A\}\wedge \\ & & \bigwedge\{p_{a\vee b} \leftrightarrow p_a \vee p_b : (a, b) \in D\}] \to p_s \end{array}$$

Note that if $D = A^2$, then $\alpha(A, D) = \chi(A)$.

$(\wedge, \to)$-canonical formulas

Theorem. Let A be a finite s.i. Heyting algebra, $D \subseteq A^2$, and B a Heyting algebra. Then

$B \not\models \alpha(A, D)$ iff there is a homomorphic image C of B and an $(\wedge, \to, 0)$-embedding $h : A \rightarrowtail C$ such that $h(a \vee b) = h(a) \vee h(b)$ for each $(a, b) \in D$.

Theorem (G.B and N.B., 2009). Every variety of Heyting algebras is axiomatized by $(\wedge, \to, 0)$-canonical formulas.

We show that for each formula φ there exist finitely many $A_1, \ldots, A_m$ and $D_i \subseteq A_i^2$ such that

$$\mathbf{IPC} + \varphi = \mathbf{IPC} + \alpha(A_1, D_1) + \cdots + \alpha(A_m, D_m)$$

Duality dictionary in the finite case

Heyting algebras	posets
s.i. Heyting algebras	rooted posets
homomorphic images	up-sets
subalgebras	bounded morphic images
Jankov formulas	de Jongh formulas
$(\wedge, \to, 0)$-canonical formulas	canonical formulas

Subframe formulas

$\alpha(A, A^2) = \chi(A)$.

$\alpha(A, \emptyset)$ is called a subframe formula.

Subframes play the same role here as submodels in model theory.

Theorem. Let A be a finite s.i. algebra and X_A its dual space. A Heyting algebra B refutes $\alpha(A)$ iff X_A is a subframe X_B.

$(\wedge, \rightarrow)$-embeddability means that we take subframes of the dual space.

There are continuum many logics axiomatized by such formulas.

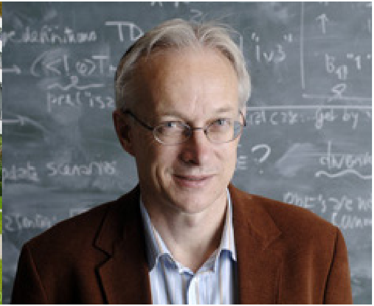
All subframe logics have the finite model property.

Zakharyaschev showed that subframe formulas are equivalent to $(\wedge, \rightarrow)$-formulas.

NNIL-formulas

NNIL-formulas are propositional formulas that do not allow nesting of implication to the left (Visser, van Benthem, de Jongh, Renardel de Lavalette, 1995)

Albert Visser



Johan van Benthem

NNIL-formulas

Gerard R. Renardel de Lavalette

NNIL-formulas and subframe formulas

Theorem (V, vB, dJ, RdL, 1995). NNIL formulas are exactly those intuitionistic formulas that are preserved under submodels.

Theorem. (de Jongh, N.B., 2006). NNIL formulas are (semantically) equivalent to subframe formulas.

$(\wedge, \vee)$-canonical formulas

We can also develop the theory of $(\wedge, \vee)$-canonical formulas $\gamma(A, D)$ using the $\to$-free locally finite reduct of Heyting algebras.

The theory of these formulas is different than that of $(\wedge, \to)$-canonical formulas.

Theorem (G.B. and N.B., 2013). Every variety of Heyting algebras is axiomatized by $(\wedge, \vee)$-canonical formulas.

$(\wedge, \vee)$-canonical formulas

Let A be a finite s.i. Heyting algebra, let s be the second largest element of A, and let D be a subset of A^2. For each $a \in A$, introduce a new variable p_a, and set

$$\begin{aligned}
\Gamma \quad = \quad & (p_0 \leftrightarrow \bot) \wedge (p_1 \leftrightarrow \top) \wedge \\
& \bigwedge \{p_{a \wedge b} \leftrightarrow p_a \wedge p_b : a, b \in A\} \wedge \\
& \bigwedge \{p_{a \vee b} \leftrightarrow p_a \vee p_b : a, b \in A\} \wedge \\
& \bigwedge \{p_{a \to b} \leftrightarrow p_a \to p_b : (a, b) \in D\}
\end{aligned}$$

and

$$\Delta \quad = \quad \bigvee \{p_a \to p_b : a, b \in A \text{ with } a \not\leq b\}.$$

Then define the $(\wedge, \vee)$-canonical formula $\gamma(A, D)$ associated with A and D as

$$\gamma(A, D) = \Gamma \to \Delta.$$

$(\wedge, \vee)$-canonical formulas

If $D = A^2$, then $\gamma(A, D) = \chi(A)$. If $D = \emptyset$, then $\gamma(A, \emptyset) = \gamma(A)$

Theorem. Let A be a finite s.i. Heyting algebra. A Heyting algebra B refutes $\gamma(A)$ iff X_A is an order-preserving image of X_B.

These formulas, called stable formulas, are counterparts of subframe formulas.

There are continuum many logics axiomatized by stable formulas.

Recently, (de Jongh and N.B., 2014) defined ONNILLI formulas (only NNILL to the left of implication).

ONNILLI formulas are (semantically) equivalent to stable formulas.

Modal logic generalizations

Connection with filtrations

There are two standard methods for proving the finite model property for modal and intermediate logics: standard filtration and selective filtration.

Taking the $(\wedge, \to, 0)$-reduct corresponds to selective filtration.

Taking the $(\wedge, \vee, 0, 1)$-reduct corresponds to standard filtration.

Modal analogues of $(\wedge, \to, 0)$-canonical formulas for transitive modal logics (extensions of **K4**) are algebraic analogues of Zakharyaschev's canonical formulas for transitive modal logics.

Whether canonical formulas can be extended to all modal logics was left as an open problem.

Connection with filtrations

Selective filtration works well only in the transitive case.

In the non-transitive case one needs to employ standard filtration.

The approach built on algebraic understanding of the standard filtration leads to a new axiomatization of all normal modal logics via stable canonical rules.

This method already has a number of applications: gives robust proof theoretic systems of modal logic, gives a new proof of decidability of admissible rules.

All this developments originated in the works of Jankov, de Jongh and Troelstra!

Thank you Dick and Anne!