Agreement Theorems from the Perspective of Dynamic-Epistemic Logic

Olivier Roy & Cedric Dégremont

November 10, 2008
Introduction and preliminaries

Overview

1. Introduction to Agreements Theorems (AT)
2. Models of knowledge and beliefs, common priors and common knowledge
3. Three variations on the result: static, kinematic and dynamic
Overview

1. Introduction to Agreements Theorems (AT)
2. Models of knowledge and beliefs, common priors and common knowledge
3. Three variations on the result: static, kinematic and dynamic

Highlights:
- The received view:
  - ATs undermine the role of private information.
- The DEL point of view:
  - ATs show the importance of higher-order information.
  - ATs show how “static” conditioning is different from “real” belief dynamics.
The Annals of Statistics
1976, Vol. 4, No. 6, 1236-1239

AGREEING TO DISAGREE

BY ROBERT J. AUMAN

Stanford University and the Hebrew University of Jerusalem

Two people, 1 and 2, are said to have common knowledge of an event
E if both know it, 1 knows that 2 knows it, 2 knows that 1 knows is, 1
knows that 2 knows that 1 knows it, and so on.

Theorem. If two people have the same priors, and their posteriors for an
event A are common knowledge, then these posteriors are equal.

If two people have the same priors, and their posteriors for a given event A
are common knowledge, then these posteriors must be equal. This is so even
though they may base their posteriors on quite different information. In brief,
people with the same priors cannot agree to disagree.

We publish this observation with some diffidence, since once one has the ap-
propriate framework, it is mathematically trivial. Intuitively, though, it is not
quite obvious; and it is of some interest in areas in which people’s beliefs about
each other’s beliefs are of importance, such as game theory and the econom
Introduction and preliminaries

Agreement Theorems in a nutshell

\textit{The Annals of Statistics}
1976, Vol. 4, No. 6, 1236-1239

\begin{center}
AGREEING TO DISAGREE\textsuperscript{1}
\end{center}

\textbf{BY ROBERT J. AUMANN}

\textit{Stanford University and the Hebrew University of Jerusalem}

Two people, 1 and 2, are said to have \textit{common knowledge} of an event $E$ if both know it, 1 knows that 2 knows it, 2 knows that 1 knows is, 1 knows that 2 knows that 1 knows it, and so on.

\textbf{Theorem.} \textit{If two people have the same priors, and their posteriors for an event $A$ are common knowledge, then these posteriors are equal.}

If two people have the same priors, and their posteriors for a given event $A$ are common knowledge, then these posteriors must be equal. This is so even though they may base their posteriors on quite different information. In brief, people with the same priors cannot agree to disagree.

We publish this observation with some diffidence, since once one has the appropriate framework, it is mathematically trivial. Intuitively, though, it is not quite obvious; and it is of some interest in areas in which people’s beliefs about each other’s beliefs are of importance, such as some threat and the economy.
AGREEING TO DISAGREE

BY ROBERT J. AUMANN

Stanford University and the Hebrew University of Jerusalem

Two people, 1 and 2, are said to have common knowledge of an event $E$ if both know it, 1 knows that 2 knows it, 2 knows that 1 knows is, 1 knows that 2 knows that 1 knows it, and so on.

THEOREM. If two people have the same priors, and their posteriors for an event $A$ are common knowledge, then these posteriors are equal.

If two people have the same priors, and their posteriors for a given event $A$ are common knowledge, then these posteriors must be equal. This is so even though they may base their posteriors on quite different information. In brief, people with the same priors cannot agree to disagree.

We publish this observation with some diffidence, since once one has the appropriate framework, it is mathematically trivial. Intuitively, though, it is not quite obvious; and it is of some interest in areas in which people's beliefs about each other's beliefs are of importance, such as game theory and the economics of information.
Introduction and preliminaries

A short result goes a long way

- Original result: [Aumann, 1976].
- No trade theorems: [Milgrom and Stokey, 1982].
- “Dynamic” versions: [Geanakoplos and Polemarchakis, 1982]
- Qualitative generalizations: [Cave, 1983], [Bacharach, 1985].
- Network structure: [Parikh and Krasucki, 1990]
A short result goes a long way

- Original result: [Aumann, 1976].
- No trade theorems: [Milgrom and Stokey, 1982].
- “Dynamic” versions: [Geanakoplos and Polemarchakis, 1982].
- Qualitative generalizations: [Cave, 1983], [Bacharach, 1985].
- Network structure: [Parikh and Krasucki, 1990].
- Good survey: [Bonanno and Nehring, 1997].
Conclusions, the received view.

Theorem

If two people have the same priors, and their posteriors for an event $A$ are common knowledge, then these posterior are the same.
Conclusions, the received view.

**Theorem**

*If two people have the same priors, and their posteriors for an event A are common knowledge, then these posteriors are the same.*

- How important is private information? Not quite...
- How strong is the common knowledge condition? Very...
- How plausible is the common prior assumption? Debated...
Conclusions, the point of view of DEL.

Theorem

If two people have the same priors, and their posteriors for an event $A$ are common knowledge, then these posterior are the same.

- The key is higher-order information.
- One should distinguish information kinematics vs information dynamics, belief conditioning vs belief update.
Definition (Epistemic-Doxastic Model)

An *epistemic-doxastic model* \( \mathbb{M} \) is a tuple \( \langle W, I, \{ \leq_i, \sim_i \}_{i \in I} \rangle \) such that:

1. \( W \) is a finite set of states.
2. \( I \) is a finite set of agents.
3. \( \leq_i \) is a reflexive, transitive and connected plausibility ordering on \( W \).
4. There is common priors iff \( \leq_i = \leq_j \) for all \( i \) and \( j \) in \( I \).
5. \( \sim_i \) is an epistemic accessibility equivalence relation. We write \( [w]_i \) for \( \{ w' : w \sim_i w' \} \).
Definition (Epistemic-Doxastic Model)

An *epistemic-doxastic model* $\mathbb{M}$ is a tuple $\langle W, I, \{\leq_i, \sim_i\}_{i \in I} \rangle$ such that:

- $W$ is a finite set of *states*. 
Definition (Epistemic-Doxastic Model)

An *epistemic-doxastic model* $\mathcal{M}$ is a tuple $\langle W, I, \{\leq_i, \sim_i\}_{i \in I} \rangle$ such that:

- $W$ is a finite set of *states*.
- $I$ is a finite set of *agents*.

**Common priors** iff $\leq_i = \leq_j$ for all $i$ and $j$ in $I$.

$\sim_i$ is an *epistemic accessibility equivalence relation*. We write $[w]_i$ for $\{w' : w \sim_i w'\}$. See: [Board, 2004, Baltag and Smets, 2008, van Benthem, 2004, Olivier Roy & Cedric Dégremont: Agreement Theorems & Dynamic-Epistemic Logic, 2009].
Definition (Epistemic-Doxastic Model)

An epistemic-doxastic model $\mathcal{M}$ is a tuple $\langle W, I, \{\leq_i, \sim_i\}_{i \in I} \rangle$ such that:

- $W$ is a finite set of states.
- $I$ is a finite set of agents.
- $\leq_i$ is a reflexive, transitive and connected plausibility ordering on $W$.
  - There is common priors iff $\leq_i = \leq_j$ for all $i$ and $j$ in $I$. 
- $\sim_i$ is an epistemic accessibility equivalence relation. We write $[w]_i$ for $\{w' : w \sim_i w'\}$. 

Definition (Epistemic-Doxastic Model)

An *epistemic-doxastic model* \( \mathcal{M} \) is a tuple \( \langle W, I, \{\leq_i, \sim_i\}_{i \in I} \rangle \) such that:

- \( W \) is a finite set of *states*.
- \( I \) is a finite set of *agents*.
- \( \leq_i \) is a reflexive, transitive and connected *plausibility ordering* on \( W \).
  - There is *common priors* iff \( \leq_i = \leq_j \) for all \( i \) and \( j \) in \( I \).
- \( \sim_i \) is an *epistemic accessibility equivalence relation*. We write \( [w]_i \) for \( \{w' : w \sim_i w'\} \).

See: [Board, 2004, Baltag and Smets, 2008, van Benthem, ]
Key notions:

▶ Knowledge: $w \models K_i \varphi$ iff $w' \models \varphi$ for all $w' \sim_i w$. 
Key notions:

- **Knowledge:** \( w \models K_i \varphi \) iff \( w' \models \varphi \) for all \( w' \sim_i w \).

- **Everybody knows:** \( w \models E_I \varphi \) iff \( w' \models K_1 \varphi \land \ldots \land K_n \varphi \) for \( 1, \ldots n \in I \).
Key notions:

▶ Knowledge: \( w \models K_i \varphi \) iff \( w' \models \varphi \) for all \( w' \sim_i w \).

▶ Everybody knows: \( w \models E_l \varphi \) iff \( w' \models K_1 \varphi \land ... \land K_n \varphi \) for \( 1, ... n \in l \).

▶ Common knowledge: \( w \models CK_l \varphi \) iff \( w' \models E_l \varphi \) and \( w' \models E_l E_l \varphi \) and...
Key notions:

- **Knowledge:** $w \models K_i \varphi$ iff $w' \models \varphi$ for all $w' \sim_i w$.
- **Everybody knows:** $w \models E_I \varphi$ iff $w' \models K_1 \varphi \land \ldots \land K_n \varphi$ for $1, \ldots, n \in I$.
- **Common knowledge:** $w \models CK_I \varphi$ iff $w' \models E_I \varphi$ and $w' \models E_I E_I \varphi$ and...
- **Beliefs:** $w \models B_i \psi \varphi$ iff $w' \models \varphi$ for all $w'$ in $\max_{\leq i}([w]_i \cap ||\psi||)$. We write $B_i \varphi$ for $B_i^\top \varphi$. 
Theorem (Static agreement)

For any epistemic-doxastic model $\mathcal{M}$ with common priors, for all $w$ we have that

$$w \not\models CK_i(B_i(E) \land \neg B_j(E))$$

where $E \subseteq W$. 

The key property: Sure-thing principle

If, first, you would believe, conditional on the fact that it is cloudy, that it will rain and,
The key property: **Sure-thing principle**

*If, first, you would believe, conditional on the fact that it is cloudy, that it will rain and, second, you would believe, conditional on the fact that it is not cloudy, that it will rain, then*
The key property: **Sure-thing principle**

_If, first, you would believe, conditional on the fact that it is cloudy, that it will rain and, second, you would believe, conditional on the fact that it is not cloudy, that it will rain, then you unconditionally believe that it will rain._

[Savage, 1954, Bacharach, 1985]
The point of view of Dynamic-Epistemic Logic

The key property: **Sure-thing principle**

![Diagram showing Agent 1's IP: P₁ and P₂ connected by a red line. Agent 2's IP: Q₁, Q₂, and Q₃ connected by a blue line.](image-url)
The point of view of Dynamic-Epistemic Logic

The key property: **Sure-thing principle**

Agent 1’s IP

\[ \text{not } B_1(E/P_1) \quad \text{not } B_1(E/P_2) \]

Agent 2’s IP

\[ B_2(E/Q_1) \quad B_2(E/Q_2) \quad B_3(E/Q_3) \]
The key property: **Sure-thing principle**

By the Sure-Thing Principle

<table>
<thead>
<tr>
<th>W</th>
<th>The CK cell</th>
<th>not $B_1(E)$</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>W</th>
<th>The CK cell</th>
<th>$B_2(E)$</th>
</tr>
</thead>
</table>
The key property: **Sure-thing principle**

The CK cell

\[ W \text{ not } B_1(E) \]

\[ W \equiv \]

\[ W \]

The CK cell

\[ B_2(E) \]

Hence no common priors
Lesson:

- They might not have the same (first-order) information, but...

What is common knowledge is precisely where their higher-order information coincide.

The cornerstone is higher-order information.

Question:

- How can CK obtain with respect to each others' beliefs?

- "Dialogues" or repeated announcements.

[Geanakoplos and Polemarchakis, 1982, Bacharach, 1985]
Lesson:

- They might not have the same (first-order) information, but... what is common knowledge is precisely where their (higher-order) information coincide.

The cornerstone is higher-order information.
Lesson:
- They might not have the same (first-order) information, but... what is common knowledge is precisely where their (higher-order) information coincide.

The cornerstone is higher-order information.

Question:
- How can CK obtain with respect to each others’ beliefs?
Lesson:

- They might not have the same (first-order) information, but... what is common knowledge is precisely where their (higher-order) information coincide.

The cornerstone is higher-order information.

Question:

- How can CK obtain with respect to each others’ beliefs?
  - “Dialogues” or repeated announcements.
    [Geanakoplos and Polemarchakis, 1982, Bacharach, 1985]
Towards kinematic agreement

Definition

A *kinematic dialogue about* $A$ is an epistemic-doxastic model $\mathcal{M} = \langle W, I, \{\leq_i, \{\sim_n,i\}_{n \in \mathbb{N}}\}_{i \in I}\rangle$ with

\[\mathcal{M} = \langle W, I, \{\leq_i, \{\sim_n,i\}_{n \in \mathbb{N}}\}_{i \in I}\rangle\text{ with}\]
Towards kinematic agreement

Definition

A *kinematic dialogue about* $A$ is an epistemic-doxastic model $\mathcal{M} = \langle W, I, \{\leq_i, \{\sim_{n,i}\}_{n \in \mathbb{N}}\}_{i \in I} \rangle$ with, for all $i \in I$, the sequence of epistemic accessibility relation $\{\sim_{n,i}\}_{n \in \mathbb{N}}$ is inductively defined as follows (for 2 agents).
Towards kinematic agreement

Definition
A **kinematic dialogue about** $A$ is an epistemic-doxtastic model
$\mathcal{M} = \langle W, I, \{\leq_i, \{\sim_n, i\}_{n \in \mathbb{N}}\}_{i \in I} \rangle$ with, for all $i \in I$, the sequence
of epistemic accessibility relation $\{\sim_n, i\}_{n \in \mathbb{N}}$ is inductively defined
as follows (for 2 agents).

$\sim_0, i$ is a given epistemic accessibility relation.
Towards kinematic agreement

Definition
A kinematic dialogue about $A$ is an epistemic-doxastic model $M = \langle W, I, \{\leq_i, \{\sim_{n,i}\}_{n \in \mathbb{N}}\}_{i \in I} \rangle$ with, for all $i \in I$, the sequence of epistemic accessibility relation $\{\sim_{n,i}\}_{n \in \mathbb{N}}$ is inductively defined as follows (for 2 agents).

- $\sim_{0,i}$ is a given epistemic accessibility relation.
- for all $w \in W$:

$$[w]_{n+1,i} = [w]_{n,i} \cap \begin{cases} B_n(A) & \text{if } w \models B_{n,j}(A) \\ \neg B_{n,j}(A) & \text{otherwise.} \end{cases}$$

with $B_{n,j}(A) = \{w' : \max_{\leq j}[w']_{n,j} \subseteq A\}$.

Intuition: $B_{n+1,i} \varphi \Leftrightarrow B^B_{n,i} \varphi \varphi$. 

Olivier Roy & Cedric Dégremont: Agreement Theorems & Dynamic-Epistemic Logic, 13
Lemma (Fixed-point)

Every kinematic dialogue about A has a fixed-point, i.e. there is a \( n^* \) such that

\[
[w]_{n^*,i} = [w]_{n^*+1,i}
\]

for all \( w \) and \( i \).

Lemma (Common knowledge)

The posteriors beliefs at the fixed-point of a kinematic dialogue are common knowledge.

Theorem (Kinematic agreement)

For any kinematic dialogue about A, if there is common priors then at the fixed-point either all agents believe that A or they all don’t believe that A.
Lemma (Fixed-point)

Every kinematic dialogue about A has a fixed-point, i.e. there is a $n^*$ such that

$$[w]_{n^*,i} = [w]_{n^*+1,i}$$

for all $w$ and $i$.

Lemma (Common knowledge)

The posteriors beliefs at the fixed-point of a kinematic dialogue are common knowledge.
Lemma (Fixed-point)

Every kinematic dialogue about A has a fixed-point, i.e. there is a $n^*$ such that

$$[w]_{n^*,i} = [w]_{n^*+1,i}$$

for all $w$ and $i$.

Lemma (Common knowledge)

The posteriors beliefs at the fixed-point of a kinematic dialogue are common knowledge.

Theorem (Kinematic agreement)

For any kinematic dialogue about A, if there is common priors then at the fixed-point either all agents believe that A or they all don’t believe that A.
Lesson:

- Common knowledge arise from “dialogues”...
Lesson:

▶ Common knowledge arise from “dialogues”...

Warnings from DEL:

▶ This is only a kinematic (i.e. conditioning) dialogue: the truth of $A$ is fixed during the process.
Lesson:

- Common knowledge arise from “dialogues”...

Warnings from DEL:

- This is only a *kinematic* (i.e. conditioning) dialogue: the truth of $A$ is *fixed* during the process.

- In general, this is not the case. $A$ might be about the agents’ information.
Lesson:

- Common knowledge arise from “dialogues”...

Warnings from DEL:

- This is only a **kinematic** (i.e. conditioning) dialogue: the truth of $A$ is **fixed** during the process.

- In general, this is not the case. $A$ might be about the agents’ information.

- Another way to look at it:

  \[
  \text{kinematic agreement} = \text{“virtual” agreement}
  \]
Towards dynamic agreement

Definition

A dynamic dialogue about $A$ is sequence of epistemic-doxastic pointed models $\{(M_n, w)\}_{n \in \mathbb{N}}$ such that:

- $M_0$ is a given epistemic-doxastic model.
- $M_{n+1} = \langle W_{n+1}, I, \leq_{n+1,i}, \sim_{n+1,i} \rangle$ with
  - $W_{n+1} = \{ w' \in W_n : w' \models B_n(A_n) \}$ with:
    - $A_n = \{ w' \in W_n : w' \models A \}$
    - $B_n(A_n)$ is $B_{n,i}(A_n) \land B_{n,j}(A_n)$ if $w \models B_{n,i}(A_n) \land B_{n,j}(A_n)$, etc...
  - $\leq_{n+1,i}, \sim_{n+1,i}$ are the restrictions of $\leq_{n,i}$ and $\sim_{n,i}$ to $W_{n+1}$.

Intuition: $B_{n+1,i} \varphi \iff [B_n \varphi!]B_{n,i} \varphi$
Lemma (Fixed-point)

Every dynamic dialogue about A has a fixed-point, i.e. there is a $n^*$ such that:

$$M_{n^*} = M_{n^*+1}$$

Lemma (Common knowledge)

The posteriors beliefs at the fixed-point of a dynamic dialogue are common knowledge.

Theorem (Dynamic agreement)

For any dynamic dialogue about A, if there is common priors then at the fixed-point $n^*$ either all agents believe that $A_{n^*}$ or they all don’t believe that $A_{n^*}$.
Lemma (Fixed-point)
Every dynamic dialogue about A has a fixed-point, i.e. there is a $n^*$ such that:

$$M_{n^*} = M_{n^*+1}$$

Lemma (Common knowledge)
The posteriors beliefs at the fixed-point of a dynamic dialogue are common knowledge.

Theorem (Dynamic agreement)
For any dynamic dialogue about A, if there is common priors then at the fixed-point $n^*$ either all agents believe that $A_{n^*}$ or they all don’t believe that $A_{n^*}$.

But...
Kinematic agreements can be different from dynamic agreements.
Kinematic agreements can be different from dynamic agreements.

\[ W\]

\[ w_1 \]

\[ p \]

\[ w_2 \]

\[ \text{not } p \]

\[ 1 \]

\[ 1 \]

\[ 2 \]
Kinematic agreements can be different from dynamic agreements.

Let \( A = p \land \neg B_2 p \)
The point of view of Dynamic-Epistemic Logic

Kinematic agreements can be different from dynamic agreements.

Let $A = p \land \neg B_2 p$

At the fixed points for the kinematic and the dynamic dialogue about $A$, we have that $[w_1]_{n^*,1} = [w_1]_{n^*,2} = \{w_1\}$
Kinematic agreements can be different from dynamic agreements.

Let $A = p \land \neg B_2 p$

At the fixed points for the kinematic and the dynamic dialogue about $A$, we have that $[w_1]_{n^*,1} = [w_1]_{n^*,2} = \{w_1\}$

Kinematic beliefs: $w \models B_{n^*,i}(p \land \neg B_{0,2} p)$, for $i = 1, 2$.

Dynamic beliefs: $w \models \neg B_{n^*,i}(p \land \neg B_{n^*,2} p)$, for $i = 1, 2$. 
Agreements theorems:
- Undermine the role of private information?
Agreements theorems:

- Undermine the role of private information?
- A better way to look at it ("DEL methodology"):  
  - Rest on higher-order information and its role in interactive reasoning.
  - Highlight the difference between belief kinematics and belief dynamics.

(Hopefully not so distant) future work:

- General (countable) case?
- Announcements of reasons and not only of opinions?
- Relaxing the common prior assumption? Agreements on everything?
Agreeing to disagree.

Some extensions of a claim of aumann in an axiomatic model of knowledge.

A qualitative theory of dynamic interactive belief revision.
In Bonanno, G., van der Hoek, W., and Wooldridge, M., editors, *Logic and the Foundation of Game and Decision Theory (LOFT7)*, volume 3 of *Texts in Logic and Games*, pages 13–60. Amsterdam University Press.

Dynamic Interactive Epistemology.

Agreeing to disagree: a survey.
Some of the material in this paper was published in [Bonanno and Nehring, 1999].

How to make sense of the common prior assumption under incomplete information.

Learning to agree.

We can’t disagree forever.
Cowles Foundation Discussion Papers 639, Cowles Foundation, Yale University.

Information, Trade, and Common Knowledge.

Communication, consensus, and knowledge.

*The Foundations of Statistics*.

van Benthem, J.
Logical dynamics of information and interaction.
Manuscript.