Is there still Logic in Bolzano's Key?

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1. My encounters with Bernard Bolzano

I am not a Bolzano scholar, but among practising logicians, my encounters with his work seem above average. The first of these was in my student days, when reading some history of logic on my own in the great works by Bocheński, and the Kneales. I was intrigued by finding how my standard textbooks had crafted an eschatological history of the field, with latter-day saints like Tarski, relegating earlier pioneers that do not fit the story line to oblivion. Later, I read my first real sample of Bolzano as a mathematician, viz. *Paradoxes of Infinity*, out of an interest in pre-Cantorian science in the making. Admittedly, not an easy read – as is true for all his texts I have seen. Bolzano as a philosopher entered my life around 1980, when looking for topics for a joint course with my more continentally trained philosophical colleague Detlev Pätzold. We settled eventually on the accounts of propositions in Leibniz, Hegel, Bolzano, and Frege – and a nice course it was! And then, at last, I was ready for the heavy bulk of the *Wissenschaftslehre*. This was partly through an interest in interfaces between logic and methodology of science, and partly as an aspiring radical in logic, intrigued by a book which is about logic, but with a quite different agenda from the modern one. My documented reaction to this reading is in van Benthem 1984, 1985. Our final encounter took place a few years ago in Prague, after wandering through a large cemetery with my colleague Eva Hajicova – trying to locate Bolzano's grave among overgrown paths, while talking about logical inference and intelligent search to a Dutch TV crew following us. We did reach the target.

2. The agenda of logic

Why look back now? Let me start by stating my non-historian's view of the modern history of logic. Like many scientific disciplines, logic flourishes while being ill-defined. Despite textbook orthodoxy, the issue what logic should be *about* is a legitimate topic of discussion, and one to which answers have varied historically. One key topic is reasoning: its valid laws for competent users, and perhaps also its sins: mistakes and fallacies. But the modern core also includes independent concerns such as formal languages, their semantic meaning and expressive power. Moreover, the modern research literature, much of it still in a pre-textbook stage, reveals a wide range of topics beyond reasoning and meaning, dealing with general structures in *information*, and many-agent activities other than reasoning, such as belief revision or *communication*. Thus, the agenda of logic keeps evolving, as it should. In this light, going back to the pioneers is not just a matter of piety, but also of self-interest.

One striking feature of older literature is its combination of issues in logic with general *methodology* of science. One sees this with Bolzano, Mill, or Peirce, but also with major modern authors, such as Tarski, Carnap, or Hintikka. The border line between logic and philosophy of science seems arbitrary. Why have 'confirmation', 'verisimilitude', or 'theory structure' become preserves for philosophers of science, and not for logicians? This separation seems an accidental feature of a historical move, viz. Frege's 'contraction of concerns', which tied up logic closely with the foundations of mathematics, and narrowed the agenda of the field to a point where fundamentalists would say that logic is the mathematics of formal systems. Admittedly, narrowing an agenda and focusing a field may be hugely beneficial. Frege's move prepared the ground for the golden age of logic in the interbellum, which produced the core logic curriculum we teach today. At the same time, broader interests from traditional logic migrated, and took refuge in other disciplines. But as its scientific environment evolved in the 20th century, logic became subject to other influences than mathematics and philosophy, such as linguistics, computer science, AI, and to a lesser degree, cognitive psychology and other experimental disciplines.

Compared with Frege, Bolzano's intellectual range is broad, encompassing general philosophy, mathematics, and logic. This intellectual span fits the above picture. Even so, I am not going to make Bolzano a spokesman for any particular modern agenda. The current professional discussion speaks for itself. But I do want to review some of his themes as to contemporary relevance. Incidentally, the main sources for the analysis in my 1985 paper, besides reading Bolzano himself, have been Kneale & Kneale 1962, and Berg 1963. After the Vienna meeting this autumn of 2002, I learnt about Rusnock 2000, whose logic chapters turned out sophisticated and congenial.

3. A short summary of Bolzanian themes

We quickly enumerate those points in Bolzano's logical system that are the most unusual and intriguing to logicians. These will return at lower speed in later sections.

- The systematic idea of decomposing propositions into general *constituents* is linguistically attractive, and reminiscent of abstract analyses of constituent structure in categorial grammars (Buszkowski 1997, Moortgat 1997, van Benthem 1991).
- In doing so, looking at different ways of setting the boundary between *fixed* and *variable vocabulary* in judging the validity of an inference is another innovation, which ties up with the recurrent issue of the boundaries of 'logicality'.
- Moving to logical core business, acknowledging different styles of reasoning: 'deducibility', 'strict deducibility', or statistical inference, each with their own merits, is a noteworthy enterprise quite superior to unreflected assumptions of uniformity.
- As to detailed proposals, consider Bolzano's central notion of deducibility. It says that an inference from premises ϕ to a conclusion ψ is valid, given a variable vocabulary A (written hence-

forth as $\phi \Rightarrow_A \psi$) if (a) every substitution instance of the A's which makes all premises true also makes the conclusion true, and (b) the premises must be *consistent*. Clause (a) is like modern validity, modulo the different semantic machinery, but with a proviso (b) turning this into a non-monotonic logic, the hot topic of the 1980s. Moreover, the role of the vocabulary argument A making inference into a ternary relation really, will also turn out significant later.

- But also other notions of inference are reminiscent of modern proposals trying to get more diversity into how people deal with large sets of data, such as 'strict deducibility': using just the minimal set of premises to get a given conclusion.
- Bolzano's statistical varieties of inference involve counting numbers of substitutions that make a given statement true. Such connections between qualitative logic and quantitative probability were still alive in Carnap's inductive logic, a fringe topic at the time – but they are coming back in force in modern logic, too.
- Very striking to logicians at the interface with AI is Bolzano's formulation of systematic properties of his notions of inference, such as versions of transitivity or the deduction theorem, some depending on the fixed/variable constituent distinction. No truth tables, model-theoretic semantics, and their ilk, but instead, some of the more sophisticated structural theory of inference that came in fashion in the 1980s.

All these themes do, or should, occur in modern logic! Let's take them up now one by one.

4. Charting the natural styles of reasoning

4.1. Different styles

Is there one notion of logical inference, or many? Mainstream logic has suggested the first, while critics often claim the second. Noticeable examples are the philosophical polemic Toulmin 1957 claiming that logical rules vary across reasoning tasks (mathematics, law, everyday life), and the psychological classic Wason & Johnson-Laird 1972, which shows how experimental subjects (or logicians off duty) vary their reasoning according to presentation and content of premises, plus the task at hand. Indeed, some diversity also occurs inside the logical heartland in constructive mathematics, or quantum mechanics (Haack 1996). But it became more prominent under the influence of AI around 1980 (McCarthy 1980 is a good source), when it turned out that problem solving and common sense reasoning come in different genres which admit of exact logical study. There is no generally accepted classification of the major styles of reasoning, but many readers will have heard of constructive logics, default reasoning, abduction, paraconsistent logic, linear logic, or logical systems of qualitative probability. Another source for variety of reasoning styles is linguistics. When analyzing the mechanisms that make us understand sentences and discourse, various logical subsystems emerge (cf. Thomason 1997).

Now that styles are a legitimate topics of study, people are going back to logicians outside of the mainstream for inspiration, and perhaps a pedigree. This is, e.g., how C.S. Peirce has come to be rediscovered by logicians (cf. Flach & Kakas 2000) – though his return as an icon of the field is dubious, as he seems firmly in the hands of the semioticists who took hold of him when he dropped by Frege's wayside.

4.2. Deviant structural rules

One striking feature of new styles of reasoning are their deviant structural rules. E.g., many styles lack the classical feature of *monotonicity*, which says that adding further premises to a valid inference does not invalidate it:

if $\phi \Rightarrow \xi$, then $(\phi \wedge \psi) \Rightarrow \xi$

Typically, non-classical styles of reasoning are non-monotonic. As a crude example, Bolzano's deducibility may hold with $\phi \Rightarrow_A \psi$, while

$\phi, \neg\phi \Rightarrow_A \psi$ is false because of the consistency clause. Now, saying that a style of reasoning is non-monotonic does not tell us much. More interesting is that there are usually positive substitutes which remain valid. A ubiquitous example is so-called 'cautious monotonicity':

if $\phi \Rightarrow \psi$ and $\phi \Rightarrow \xi$, then $(\phi \wedge \psi) \Rightarrow \xi$

This property also holds for Bolzano's deducibility.

4.3. Mechanisms

Failures of classical structural properties like monotonicity are just symptoms of some underlying cause of broader logical interest. We want a proper diagnosis of the underlying *mechanisms* that produce the varieties of reasoning! – Van Benthem 1989 makes a connection at this point with something called 'Bolzano's Program'. – We do not just switch from classical logic on Sundays to default logic on weekdays: there must be systematic factors at work. There is no consensus among logicians, or agreed system for these mechanisms. General factors behind modern styles of reasoning have to do with issues like

- dealing with *contradictory information*
- keeping track of computational *resources* in inference
- the *dynamics* of successive stages of communication, and
- the interplay of *preferences* of interacting agents

4.4. Architecture

Here is a final broad issue raised by all this. If we have the ability to engage in all these different styles, we must be able to integrate information from different sources obtained by different mechanisms. Thus, the modular *logical architecture* of our over-all reasoning system becomes a new concern, which was still absent when there seemed just one logical reasoning to begin with. Gabbay 1998 has very interesting relevant ideas on 'combining logics' – but it seems fair to say that no stable paradigm has yet emerged. Getting clear on the variety and architecture of reasoning styles is a major challenge. But it is not one where Bolzano as I read him has much to offer – except perhaps giving some blessing to the enterprise.

5. Structural rules as a logical bottom level

To the receptive reader, Bolzano's ideas also lead to more specific issues beyond our general ideologizing so far! In this section and the next, we discuss a few, from more concrete to more speculative.

5.1. Structural rules once more

Bolzano did not offer us a theory of the logical constants, the way standard logic tells the story of Boolean operators or first-order quantifiers. But is any significant logical content left at the level of just his rules? The answer is positive. Indeed, Bolzano himself gave examples of what one would nowadays call *structural rules*: namely, general properties of an abstract inference relation $X \Rightarrow C$ between finite sequences of propositions X and single propositions C. Here are some examples of such rules for classical inference. We formulate them in a format which does not assume anything about the structure of the premises, except that they form a sequence of propositions.

$C \Rightarrow C$	Reflexivity
if $X, Y \Rightarrow C$, then $X, P, Y \Rightarrow C$	Monotonicity
if $X \Rightarrow D$ & $Y, D, Z \Rightarrow C$, then $Y, X, Z \Rightarrow C$	Cut Rule
if $X, P_1, P_2, Y \Rightarrow C$, then $X, P_2, P_1, Y \Rightarrow C$	Permutation
if $X, P, Y, P, Z \Rightarrow C$, then $X, P, Y, Z \Rightarrow C$	Right Contraction
if $X, P, Y, P, Z \Rightarrow C$, then $X, Y, P, Z \Rightarrow C$	Left Contraction

Over the years, a host of "characterization theorems" has been found for non-standard varieties of consequence. We list some examples.

5.2. Classical inference

We start with the familiar Tarskian logical consequence.

> **Theorem**: An abstract inference relation $X \Rightarrow C$ can be represented as classical set inclusion between sets of models for its premises X and its conclusion C if and only if it satisfies Reflexivity, Monotonicity, Cut, and Contraction.

The simple proof is in van Benthem 1996, Chapter 7. This result shows that non-classical styles of reasoning must break at least one of these laws. As Reflexivity, Cut, and Contraction seem basic, this explains the ubiquity of non-monotonicity.

5.3. Dynamic inference

Other results of this kind uncover other mechanisms of reasoning. Here is a case from the same book. In *update logics* for communication, propositions are operations which transform information states. Valid dynamic consequence for a sequent $X \Rightarrow C$ then says that the successive transformations for the premises in the sequence X always reach a state where an update with the conclusion C has no further effect. It is easy to see that this style of reasoning loses most of the above structural rules. But it has the following valid substitutes:

if $X \Rightarrow C$, then $A, X \Rightarrow C$	Left-Monotonicity
if $X \Rightarrow A$ and $X, A, Y \Rightarrow C$, then $X, Y \Rightarrow C$	Left-Cut
if $X \Rightarrow A$ and $X, Y \Rightarrow C$, then $X, A, Y \Rightarrow C$	Cautious Monotonicity

> Theorem: An abstract inference relation $X \Rightarrow C$ can be represented as dynamic inference for propositions transforming information states if and only if it satisfies Left-Monotonicity, Left-Cut, and Cautious Monotonicity.

For a more elaborate presentation, cf. van Benthem 2003.

5.4. Preferential inference

Our last example concerns reasoning over a structured universe of models, ordered by some *preference relation* as to plausibility or simplicity: the proper setting for default rules in AI and natural language. With this style, a conclusion C must hold, not in all, but only in *all most preferred* models of the premises X. This time, we show another style of analysis, also involving logical constants. The following principles clearly hold for preferential inference:

$C \Rightarrow C$	Reflexivity
if $X \Rightarrow C_i$ for each $i \in I$, then $X \Rightarrow \wedge_i C_i$	Conjunction
if $X \Rightarrow C$, then $X \Rightarrow C \vee D$	Weakening

Taken together, these imply that each proposition A has some strongest conclusion, say $best(A)$. The next valid principle is also well-known for conditional reasoning:

if $A_i \Rightarrow C$ for each $i \in I$, then $\vee_i A_i \Rightarrow C$ Disjunction

The final principle that we need uses the above 'best conclusions':

if $\vee_i A_i \Rightarrow C$, then $\wedge_i best(A_i) \Rightarrow C$ 'Best of All'

Van Benthem 1989 proves that these principles suffice for representing any inference relation as preferential consequence over some suitably ordered universe.

5.5. What about Bolzano's deducibility?

In van Benthem 1985, a similar line was taken on Bolzano's deducibility, including a bunch of valid structural rules. Let's finish the job here. The representation argument itself will tell us the structural requirements. Recall that deducibility says that there is at least one model for all premises together, and that all such models are also models for the conclusion. Now, consider any abstract inference relation $X \Rightarrow C$. For each proposition A, let

$$\#(A) =_{def} \{X \mid X \Rightarrow A\}$$

It suffices to show that the following equivalence holds:

$$X_1,\ldots, X_k \Rightarrow C \text{ iff (a) } \cap_i \#(X_i) \subseteq \#(C)\text{, and (b) } \cap_i \#(X_i) \text{ is non-empty}$$

From right to left, by (b), we must have a sequence of propositions U such that

$$U \Rightarrow X_i \text{ for each } i \in I$$

Now, on the basis of our intended interpretation, the following holds

$$\text{if } U \Rightarrow X_i \text{ for each } i \in I\text{, then } X_1,\ldots, X_k \Rightarrow X_i \quad \text{Infimum}$$

Therefore, if we adopt this, by (a), $X_1,\ldots, X_k \Rightarrow C$, which was what we need. Now for the converse. Suppose that $X_1,\ldots, X_k \Rightarrow C$. Again by our interpretation, the following holds:

$$\text{if } X_1,\ldots, X_k \Rightarrow C\text{, then } X_1,\ldots, X_k \Rightarrow X_i \qquad \text{Consistency}$$

This takes care of clause (b). Now for clause (a). Let $U \Rightarrow X_i$ for each $i \in I$. We need to have that $U \Rightarrow C$. We could derive this from $X_1,\ldots, X_k \Rightarrow C$, provided that we have Cut. But we do not: as the latter structural rule fails for deducibility. E.g., $p, \neg q \Rightarrow p$, and $p, q \Rightarrow q$ in Bolzano's sense. But $p, \neg q, q \Rightarrow q$ is not true. Therefore, only weaker Cut variants can hold. One valid fix is 'Simultaneous Cut':

$$\text{if } X_1,\ldots, X_k \Rightarrow C \text{ and } U \Rightarrow X_i \text{ for each } i \in I\text{, then } U \Rightarrow C$$

Theorem: Bolzano's deducibility is completely characterized by the rules of Infimum, Consistency, and Simultaneous Cut.

We can gift-wrap this a little and make the result look more profound by hiding these proof-generated structural rules (a trade secret of the literature), but this will do here. For a general theory of structural rules and varieties of inference, cf. Restall 2000.

6. Further issues about reasoning

Finding natural packages of structural rules for use with specific reasoning tasks is not the only logical theme in a Bolzanian mode! Here are some further issues – many still in their infancy – that arise naturally when thinking about styles of reasoning.

6.1. Computational complexity

A striking feature of deducibility is its *higher complexity* than Tarskian consequence. Deducibility simply encodes satisfiability, as a formula ϕ has a model iff $\phi \Rightarrow \phi$ in Bolzano's sense. Hence, it is not effectively axiomatizable, as satisfiability is a non-axiomatizable notion. And strict deducibility is even more complex. Van Benthem 1985 has exact estimates. This demonstrates a paradox: current systems of 'realistic' reasoning tend to have higher complexity than their classical counterparts. E.g., intuitionistic propositional logic has a harder decision procedure than classical logic, default versions of first-order logic even become undecidable, and so on. Many practitioners believe the total balance of using such logics plus compact data representations allowed by them is favourable, but there is no conclusive analysis underpinning this. Being a Bolzanian pluralist has its price …

6.2. Alternative data structures?

This issue of packaging reasoning is crucial to real logical systems. Their performance depends on two factors: the inference engine plus the format of representation for propositions. Modern logic has little systematic to say about the latter aspect, and one has to look to the philosophy of science, or computer science. Given this second degree of freedom, there is even a general question to which extent deviant styles of inference can be reduced to a package using a classical inference engine with suitably varying data structures.

6.3. Architecture

One more concrete aspect of an extended reasoning style-book is how to let different styles (Tarskian, Bolzanian, etc.) *communicate*. In this sense, one does not just want separate characterization results as in Section 5, but joint systems mixing, say, classical with non-monotonic inference. This already happened with the earlier preferential reasoning, where Weakening really amounts to the pattern

$$\text{if } X \Rightarrow_{preferential} C \text{ and } C \Rightarrow_{classical} D\text{, then } X \Rightarrow_{preferential} D$$

Van Benthem 1993 presents a complete merge of classical and preferential inference just in terms of structural rules. More generally, Gabbay 1996 proposes *labelled deduction* as a way of merging annotated logical inferences from different sources.

Many further features in Bolzano's reasoning style strike a chord today, such as the juxtaposition of logical and *probabilistic reasoning*. Probabilistic themes are making their way back from philosophy of science into mainstream logic, both in inference and in semantics, including informational update. But we forego this trend here.

7. Logical and non-logical vocabulary

Sometimes, not presence, but absence of an agenda item can be refreshing. Bolzano does not emphasize the usual logical constants, making us think harder about what these are, and why they should be so important anyway. First, the idea that one has to explicitly decide on a border-line between fixed and variable vocabulary makes a lot of sense. Standard logical constants like 'not', 'and', 'or', 'for all', 'there exists' usually come with their meaning fixed – but they need not. E.g., in truth-functional propositional logic, the following inference is valid for every Boolean operator #:

$$\#\#\#\phi \Rightarrow \#\phi$$

Conversely, nothing prevents us from fixing the meaning of standard linguistic constructions like the comparative *-er*, which seems about just as "logical" in ordinary reasoning as the Booleans. Consider the following evergreen:

John is taller than Mary, Mary is taller than the iBook,
therefore John is taller than the iBook.

On Bolzano's view, we cannot say absolutely whether this is valid: it depends on what we fix in the vocabulary: {'John', 'Mary', 'the iBook', 'taller than'}. In fact, through progressive abstraction, we can dissect the inference even more slowly, down to minimal valid versions, with fixed parts indicated in black, like

x **is** *T*-**er than** *y*, *y* **is** *T*-**er than** *z*, *therefore* *x* **is** *T*-**er than** *z*

where all that is left is the Cheshire smile of our logical Cat. All this reflects a common idea in logical semantics of natural language: *logicality comes in degrees*. Boolean operations and standard first-order quantifiers have it, but so do the above comparative particles, *generalized quantifiers*, or other expressions. This reinforces the question what the privileged logical constants are anyway. There is no generally agreed answer to this, but the subject seems to be heating up, with semantic invariance approaches, as well as more proof-theoretic, or computational ones (cf. van Benthem 2002 for an up-to-date discussion of approaches and known results).

8. The interplay of inference and vocabulary

Back to concrete technical issues! Even in its general form, the fixed/variable distinction has the effect of making valid inference a *ternary*, rather than the usual binary notion. It has two propositional arguments ϕ, ψ and one A for that vocabulary in the total language of the relevant assertions whose meaning is to be kept *fixed*:

$$\phi \Rightarrow_A \psi$$

For convenience, we look at standard consequence in this perspective, without Bolzano's consistency clause.

8.1. Structural rules for ternary consequence

In this format, the earlier structural rules become more sophisticated. For instance, we must also ask what happens as we vary the vocabulary argument. Indeed it is easy to see that we have upward monotonicity:

if $\phi \Rightarrow_A \psi$ and $A \subseteq B$, then $\phi \Rightarrow_B \psi$

Also, we need to determine best versions for all earlier structural rules. For instance, they will all remain valid when we keep the vocabulary choice the same, as in

if $\phi \Rightarrow_A \psi$ and $\psi \Rightarrow_A \xi$, then $\phi \Rightarrow_A \xi$

Van Benthem 1985 also considers more ambitious combinations like:

if $\phi \Rightarrow_A \psi$ and $\phi \Rightarrow_B \psi$, then $\phi \Rightarrow_{A \cap B} \psi$

But this is invalid, witness the following counter-example. Consider the valid inference

$p \wedge q \Rightarrow p \vee T$ with T the always true proposition

Now make some more things variable. Then the following two more general versions of this inference are still valid, as is easy to see:

$p \wedge q \Rightarrow_{\{\wedge, \vee\}} p \vee T$ making the interpretation of T variable
$p \wedge q \Rightarrow_{\{T, \vee\}} p \vee T$ making the interpretation of $\wedge$ variable

Or in another notation, with # for a freely interpretable expression in its category:

both $p \wedge q \Rightarrow p \vee \#$ and $p \# q \Rightarrow p \vee T$ are valid

But taking the intersection of the fixed vocabularies, we would get

$$p \wedge q \Rightarrow_{\{\vee\}} p \vee T$$

or in the other notation:

$$p \# q \Rightarrow p \vee \#$$

But this inference is not valid, since we can now freely interpret $\wedge$ as 'always true', T as 'always false', and make p false.

This example is related with the following broader issue:

Is there always some most general valid schema to be found behind a given valid inference?

The answer is negative, at least in our Bolzanian sense, as the above gave us two minimal valid variants of a schema which do not combine to a common instance.

Combining earlier facts, we only get valid transitions like

if $\phi \Rightarrow_A \psi$, and $\psi \Rightarrow_B \xi$, then $\phi \Rightarrow_{A \cup B} \xi$

The complete structural theory of ternary consequence, even for classical logic, seems open.

8.2. Harmony of language and inference

The above suggests there should be close connections between a notion of inference among propositions and their language. There are few results on this in logic, but a striking one is the *interpolation theorem*:

> If $\phi \models \psi$, then there exists an 'interpolant' α in the intersection of the vocabularies of ϕ and ψ such that $\phi \models \alpha \models \psi$

This is often seen as a desideratum on the design of well-balanced formal languages and notions of consequence, but no systematic theory exists of even first-order entailment plus vocabulary sets. Indeed, Mason 1985 shows that the complete theory of first-order logic in this sense has a huge non-arithmetically definable complexity.

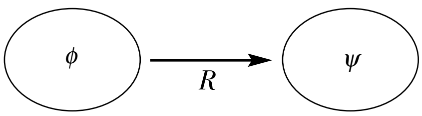
8.3. Entailment along a relation

Nevertheless, ternary notions of inference have been proposed in Barwise & van Benthem 1999, with the following motivation. The third argument A can do very useful work! Actual inference often *transfers* information about one situation to another:

> If I know ϕ about one situation **M**, then I know ψ about another situation **N**, provided it is suitably related to **M**

So let us define entailment along a relation as follows:

> If $\mathbf{M} \models \phi$ and $\mathbf{M}\ R\ \mathbf{N}$, then $\mathbf{N} \models \psi$



Taking R to be the identity, we have ordinary classical consequence, which involves no situation travel. But in general, R can be any reasonable structural relation between models: isomorphism, bisimulation, extension, etc. The central example in the cited paper lets R be some minimal relation guaranteeing that vocabulary A is invariant between $\mathbf{M}$ and $\mathbf{N}$, such as *potential isomorphism w.r.t. A*. Interpolation theorems have an A-interpolant as a conclusion from ϕ couched in terms of A, so that it can 'cross' from $\mathbf{M}$ to $\mathbf{N}$, where it implies ψ.

Van Benthem 1999 introduced a proof calculus for ternary consequence making the vocabulary A explicit in model-crossing inferences. E.g., with the above pictures, the important logical rule of implication introduction clearly fails in the form

if $\phi, \alpha \Rightarrow_A \psi$, then $\phi \Rightarrow_A \alpha \rightarrow \psi$

But the rule is valid in a more careful version, obtained by adapting the third argument:

if $\phi, \alpha \Rightarrow_A \psi$, then $\phi \Rightarrow_{A \cup voc(\alpha)} \alpha \rightarrow \psi$

This calculus has some features reminiscent of the care which Bolzano had to exercise with maintaining the right vocabulary sets.

8.4. Nonstandard inference, nonstandard language?

The intimate connection between language and inference also emerges in other ways. If we weaken classical logic, linguistic distinctions may come to the fore that call for separate connectives. E.g., linear logic has two non-equivalent conjunctions: one of 'choice' and one of 'product', and modal dependence logics have non-equivalent single and polyadic first-order quantifiers. By contrast, logical radicals are often conservative as to vocabulary. Intuitionistic logic came to do away with classical logic, but it kept its language, without adding its own 'constructive' logical operators. Likewise, non-monotonic logics use preference models, or tolerate inconsistency – but their proponents usually do not question the traditional *language*! A rare exception is the sophisticated study Belnap 1982 of complete 'display vocabularies' for substructural logics. But styles of reasoning come with their own natural language, which merits simultaneous study. Even Bolzano's own notion of deducibility invites extension to a modal language formulating the consistency of the premises as a separate explicit fact.

9. In between truth and validity

9.1. Consequence in a model

The preceding analysis has taken a perhaps anachronistic modern line on Bolzano's consequence in that, in line with modern notions, we quantified over the totality of all possible models. But one might claim that Bolzano has one World in mind, where the only thing that varies are different substitutions of real predicates for variable parts of assertions. In the latter approach, validity of an inference becomes relativized to a model, and we might want to say that there is a fourth argument **M**. Forgetting the orthogonal issue of consistent premises, we can then define a local version of consequence – this time, for convenience, letting A stand for the variable part of the vocabulary:

CONSEQUENCE IN A MODEL: $\mathbf{M} \models \phi \Rightarrow_A \psi$
every A-substitution with **M**-definable predicates which makes ϕ true in **M** also makes ψ true in **M**

This makes consequence a sort of universal second-order statement

$\forall A$: $\phi \Rightarrow \psi$

about single models **M**, with the universal quantifier $\forall A$ read with a substitution interpretation. In special cases, this can still amount to ordinary validity. E.g., the Hilbert-Bernays completeness theorem says that the valid first-order formulas in this sense on the natural numbers (with only Δ^0_2 definable predicate substitutions) are just the usual validities of first-order logic. Doets 1987 has further positive results – but there are many models for which this theory is not known.

9.2. Technical excursion

The general question here is which reasoning is supported by a model **M**, when we fix the meaning of some parts of an implication, or a statement in general, while the variable parts are allowed to run over all **M**-definable objects and predicates. Consider the infinite domain par excellence of the natural numbers. As Hilbert & Bernays showed, schematic truth of this sort equals first-order validity on the complete natural numbers $(N, <, +, x)$. But we can also look at reasoning inside weaker structures of interest, such as additive arithmetic $(N, <, +)$, in which the definable sets are just the 'semi-linear' ones – or even the underlying linear order $(N, <)$. In the latter model **N**, the only definable sets are the finite and cofinite ones. Now, we have an obvious difference with standard consequence:

FACT: Bolzano validity in **N** genuinely extends first-order validity.

To see this, consider a first-order formula $\phi(R)$ stating that R is a strict linear order without beginning or end. This formula is satisfiable in general, e.g., in the integers. But it is not satisfiable by any $<$-definable relation R^* in **N**. For, any such relation R^* must have infinitely many points below and above every number n. In particular, the set $\{n \in N \mid R^*(n, 0)\}$ will then be $<$-definable in **N**, as the point 0 is evidently $<$-definable. But this is then a $<$-definable subset of N which is neither finite nor cofinite, which contradicts the earlier observation. More generally, we can show that validity in this sense will be Π^0_1, and hence non-axiomatizable.

There is some technical interest to consequence in a model, as a question at the border line of first-order and higher-order logic. But there is also an interesting more general issue. Setting up things in this way mixes up the usual semantic perspective. A model is given which interprets only part of our language fully, while other predicates may still be freely interpreted. Think of a fairy tale which is supposed to take place on this planet, with some standard terms having their usual denotations, while others must be 'filled in'. Or of scientific theories, starting from observational vocabulary whose interpretation is anchored to reality, while the theoretical terms can range over the whole mathematical predicate structure of this model. These are the real situations in which we reason, and their heterogeneous mixtures of semantic evaluation and proof are equally a fact of life. So, if Bolzano's notions do not quite fit received methodological distinctions in logic, then this may be all to the good. Our reasoning practice does not appear to fit them either.

10. Cognitive aspects after all?

Bolzano is widely seen as the philosopher of abstract propositions, far removed from psychological blemishes. Nevertheless, many themes in this paper suggest links with the actual reasoning performed by non-Platonic humans like us. We saw this with attention to diverse styles of task-dependent reasoning, with degrees of logicality for the expressions of natural language that we actually use, with inferences transferring information across discourse situations, with global architecture of reasoning styles, or with mixtures of such neatly compartmentalized logical activities as semantic evaluation and proof. When we take all this seriously, it becomes hard not to go one step further, and do something which Frege has forbidden – but probably also Bolzano: take the psychological facts seriously. All the above topics border on cognitive science and the experimental study of human reasoning, and the eventual agenda of modern logic will also have to come to better terms with that than the by now pretty stale slogan of 'anti-psychologism'.

11. Conclusions

We have surveyed some aspects of Bolzano's logic from a modern standpoint, stressing in particular his different styles of consequence, the essential ternary nature of consequence when language is taken into account, and the mixed notion of consequence in a model. In all three cases we included some new technical observations to show that the issues are still alive. But the more general thrust is this.

Bolzano's work remains interesting for logic today, both in its general sweep, and in some of its details. Partly, it is attractive precisely *because* it is so non-mainstream, and hence valuable for modern agenda discussions. Its themes crossing logic and philosophy of science reflect current rapprochements, while its thrust also seems to fit with some themes from AI. Classical mathematical logic has had an Austrian icon in Kurt Gödel: modern logic might consider at least having a Czech-Austrian patron saint.

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