Intuitionism, Generalized Computability and Effective Operations

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In Honour of Anne Troelstra and Dick de Jongh
Joint work with Eric Faber

My mathematical pedigree

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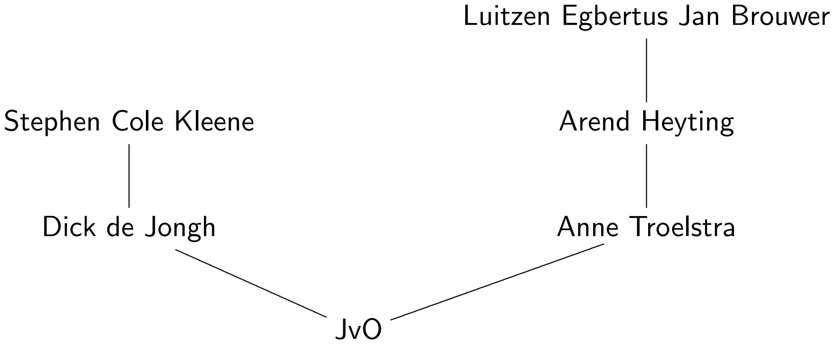

Starting point: the notion of a *Partial Combinatory Algebra* (pca). A pca is a set A together with a partial binary map $(a, b \mapsto ab)$: $A \times A \to A$. We write $ab\downarrow$ for: *ab is defined*. We also write abc for $(ab)c$.

There should be elements k and s satisfying:

$$\mathsf{k}ab = a$$

$$\mathsf{s}ab\downarrow \text{ and, if } ac(bc)\downarrow, \text{ then } \mathsf{s}abc = ac(bc)$$

Prime example: $\mathcal{K}_1$, the set of natural numbers, where nm is the result (if defined) of the n-th computable function applied to m.

Peter Johnstone calls pcas *Schönfinkel Algebras*.

Moses Ilyich (or is it *Isayevich*?) *Schönfinkel* is one of the more mysterious figures in the history of logic. He was born in 1889 (or was it 1887?) in Ukraina. He worked from 1914 (!) to 1924 under Hilbert in Göttingen, during which period one paper appeared: *Über die Bausteine der mathematischen Logik* in *Mathematische Annalen* 92, 1924. However, this paper appears to have been written by someone else, who took notes during lectures by Schönfinkel.

A second paper, coauthored by Bernays, appeared in 1927; by this time, however, Schönfinkel was already in a mental hospital in Moscow.

He died in 1942 in Moscow; his papers were used for firewood by his neighbours.

Back to pcas.
Every pca A admits:

- *pairing and unpairing combinators*: elements $\pi, \pi_0, \pi_1 \in A$ satisfying $\pi_0(\pi ab) = a$, $\pi_1(\pi ab) = b$
- *Booleans*: elements T and F and an element u satisfying $u\mathrm{T}ab = a$, $u\mathrm{F}ab = b$ (we can pronounce $uxyz$ as: if x then y else z)
- *Curry numerals*: elements $\overline{n}$ for every natural number n; for every computable function f there is an element $a_f \in A$ such that for every n, $a_f\overline{n} = \overline{f(n)}$

Every pca A gives rise to a category of *assemblies* $\mathrm{Ass}(A)$: an object of $\mathrm{Ass}(A)$ (an A-assembly) is a pair (X, E) where X is a set, and for each $x \in X$, $E(x)$ is a nonempty subset of A.

A morphism $(X, E) \to (Y, F)$ is a function $X \stackrel{f}{\to} Y$ which is *tracked* by some $b \in A$: for every $x \in X$ and every $a \in E(x)$, $ba \in F(f(x))$.

The category $\mathrm{Ass}(A)$ is cartesian closed and has a natural numbers object.

The Model HEO of Hereditarily Effective Operations

(Kreisel, Troelstra)
Consider Type $:\equiv\ 0 \,|\, \text{Type} \to \text{Type}$
The type $\underline{n}$ is $(\cdots((0 \to \underbrace{0) \to 0) \cdots \to 0}_{n \text{ times}})$

$\text{HEO}_{\underline{0}} = \mathbb{N}$; $x \equiv_{\underline{0}} y$ iff $x = y$
$\text{HEO}_{\underline{n+1}} \ = \ \{a \in \mathbb{N} \,|\, \forall xy \in \text{HEO}_{\underline{n}}(ax{\downarrow} \wedge (x \equiv_{\underline{n}} y \to ax = ay))\}$
$a \equiv_{\underline{n+1}} b$ iff $\forall x \in \text{HEO}_{\underline{n}}(ax = bx)$

In $\mathrm{Ass}(\mathcal{K}_1)$, the full type structure on $\mathcal{N} = (\mathbb{N}, \lambda x.\{x\})$ is "isomorphic" to HEO:
Let $(N_0, E_0) = \mathcal{N}$ and $(N_{n+1}, E_{n+1}) = \mathcal{N}^{(N_n, E_n)}$
The objects (N_{n+1}, E_{n+1}) can be described as follows:
$N_{n+1} = \{f : (N_n, E_n) \to$
$\mathcal{N} \mid f$ is tracked by some element of $\mathrm{HEO}_{\underline{n+1}}\}$
$E_{n+1}(f) = \{a \mid a \text{ tracks } f\}$

The category $\mathrm{Ass}(A)$ comes equipped with functors

$$\mathrm{Set} \underset{\nabla}{\overset{\Gamma}{\leftrightarrows}} \mathrm{Ass}(A)$$

where Γ is the global sections functor and ∇ sends a set X to the assembly $(X, \lambda x.A)$. We have $\Gamma \dashv \nabla$

A functor $\mathrm{Ass}(A) \to \mathrm{Ass}(B)$ is a Γ-functor if the diagram

$$\begin{array}{ccc} \mathrm{Ass}(A) & \longrightarrow & \mathrm{Ass}(B) \\ & {\scriptstyle\Gamma}\searrow \quad \swarrow{\scriptstyle\Gamma} & \\ & \mathrm{Set} & \end{array}$$

commutes.

Think of a Γ-functor as a functor which is the identity on the level of sets and functions.

Definition

(J. Longley) Given pcas A and B, an *applicative morphism* $A \to B$ is a function γ which assigns to every $a \in A$ a nonempty subset $\gamma(a)$ of B, in such a way that for some element $r \in B$ (the *realizer* of γ) the following holds:
whenever $ab\downarrow$ in A, $u \in \gamma(a)$, $v \in \gamma(b)$, we have $ruv \in \gamma(ab)$.
Composition is composition of relations. We obtain a category PCA.
Every applicative morphism $\gamma : A \to B$ determines a regular Γ-functor $\gamma^* : \mathrm{Ass}(A) \to \mathrm{Ass}(B)$: it sends (X, E) to (X, F) where $F(x) = \bigcup_{a \in E(x)} \gamma(a)$.

Example of an applicative morphism: consider $\mathcal{K}_2^{\mathrm{rec}}$, the pca of "function realizability" but restricted to the total recursive functions.

There is an applicative morphism $\mathcal{K}_2^{\mathrm{rec}} \to \mathcal{K}_1$ which sends a function to the set of its indices. The realizer simulates the action of $\mathcal{K}_2^{\mathrm{rec}}$ in $\mathcal{K}_1$.

Theorem

(J. Longley) Every regular Γ-functor $\mathrm{Ass}(A) \to \mathrm{Ass}(B)$ is of the form γ^* for some (essentially unique) $\gamma : A \to B$.

Can we characterize those applicative morphisms γ for which γ^* has a right adjoint?

Hofstra-vO: these are the *computationally dense* applicative morphisms. The definition of "computationally dense" was rather complicated.

Decidable Applicative Morphisms

Definition (Longley) An applicative morphism $\gamma : A \to B$ is *decidable* iff γ^* preserves finite coproducts (equivalently, if γ^* preserves the Natural Numbers Object)

Clearly, every computationally dense morphism is decidable.

There is, for every pca A, exactly one decidable morphism $\mathcal{K}_1 \to A$: it sends n to $\bar{n}$, the n-th Curry numeral in A.

Definition. Let $\gamma : A \to B$ be applicative. A partial endofunction f on A is *representable w.r.t.* γ if there is an element b such that, whenever $f(a) = a'$ then $b\gamma(a) \subseteq \gamma(a')$

Construction

Given a pca A and a partial endofunction f on A, there is a universal decidable morphism $\iota_f : A \to A[f]$ w.r.t. which f is representable: whenever $\gamma : A \to B$ is decidable and f is representable w.r.t. γ, then γ factors uniquely through ι_f.

The construction generalizes that of forming the pca of partial functions computable in an oracle: if $A = \mathcal{K}_1$ and f is a partial function on the natural numbers, then $A[f]$ is isomorphic to the pca of indices of functions "computable in the oracle f".

The morphism ι_f is computationally dense and induces an adjunction: $\mathrm{Ass}(A[f]) \to \mathrm{Ass}(A)$.

Extension of the construction to Type 2 effective operations

Let Tot_A be the set of total, A-computable functions $A \to A$.
For $f \in \mathrm{Tot}_A$, its set of indices is

$$I_1^A(f) = \{a \in A \mid \forall x \in A(ax = f(x))\}$$

If $\gamma : A \to B$ is an applicative morphism, we define for a partial function $f : A \to A$ its set of indices *relative to* γ:

$$I_1^\gamma(f) = \{b \in B \mid \forall a \in A(f(a)\downarrow \Rightarrow b\gamma(a) \subseteq \gamma(f(a)))\}$$

Note: if $f \in \mathrm{Tot}_A$ then $I_1^\gamma(f)$ is nonempty; but more functions may become representable w.r.t. γ.

Let A^A be the set of all total functions $A \to A$.
A *partial effective operation of type 2* on A is a partial function $F : A^A \to A$ which has a realizer in A: an element $a \in A$ such that whenever $f \in \mathrm{Tot}_A$ and $F(f)$ is defined, then $aI_1^A(f) = \{F(f)\}$.
Let $I_2^A(F)$ be the set of realizers of F in this sense.
We have a similar definition for an "effective operation relative to an applicative morphism" $\gamma : A \to B$.

Let $\gamma : A \to B$ applicative. A partial function $F : A_A^A \to A$ is a (type 2-) effective operation *relative to* γ, if there is an element $b \in B$ which realizes it:
for every $f \in A^A$ such that $F(f)$ is defined, we have
$bI_1^\gamma(f) \subseteq \gamma(F(f))$

Example

Define the "existential quantifier" functional $E : \mathrm{Tot}_A \to A$ by

$$E(f) = \begin{cases} \mathsf{T} & \text{if for some } a \in A,\ f(a) = \mathsf{T} \\ \mathsf{F} & \text{otherwise} \end{cases}$$

In e.g. $\mathcal{K}_1$, E is not an effective operation. But there is a "least" applicative morphism $\mathcal{K}_1 \to \mathcal{K}_1[E]$ relative to which it is.

Theorem

Given a pca A and a partial functional $F : A^A \to A$, there exists a pca $A[F]$ with underlying set A, and a decidable applicative morphism $\iota_F : A \to A[F]$ satisfying:

1) F is an effective operation relative to ι_F.
2) Whenever $\gamma : A \to B$ is such that F is an effective operation relative to γ, then γ factors through ι_F in a unique way.

Example (continued)

Suppose A is $\mathcal{K}_1$, F a partial functional $\mathbb{N}^{\mathbb{N}} \to \mathbb{N}$. A function $f : \mathbb{N} \to \mathbb{N}$ is representable in $A[F]$ precisely when it is $(S1 - S9)$-recursive in F.

For example, if E is the "existential quantifier" functional, the functions representable in $\mathcal{K}_1[E]$ are precisely the hyperarithmetical functions.