On the topological interpretations of provability logic

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To Anne Troestra and Dick de Jongh

Preamble: On Dick's impact

Some (but not all) of Dick's remarkable contributions:

- De Jongh Completeness Theorem for intuitionistic predicate logic w.r.t. Heyting Arithmetic
- De Jongh–Sambin fixed point theorem in provability logic
- De Jongh–Parikh theorem on well-partial orderings
- De Jongh–Veltman semantics of interpretability logics

Not all of them are well-documented.

De Jongh Completeness Theorem

Dick showed us:
The logic of Heyting Arithmetic is intuitionistic first order logic

"The logic of . . . is . . ." signifies a change from the more traditional perception of the role of logic in the foundations of mathematics.

A chicken or egg question: Is Mathematics built up on top of Logic, or is Logic an abstraction recovered from an independently existing mathematical reality?

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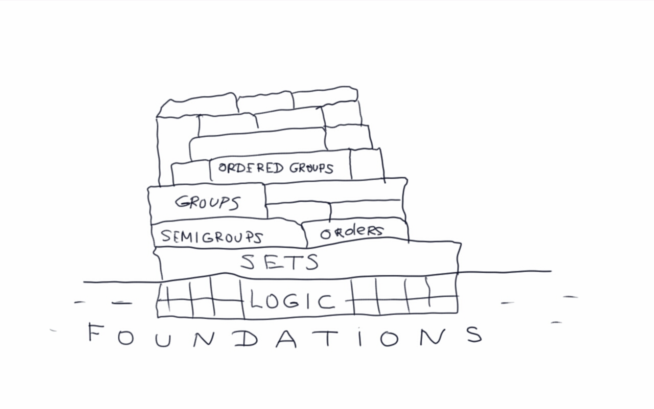
Bourbakism: Mathematics is like a building constructed axiomatically from below. At the bottom there is pure logic, then the ascetic world of sets. On top of it there are simple structures, higher up are richer and more complex structures.



Anti-bourbakism: The world of mathematical structures is not built up, but already exists somewhere. The laws of logic represent its most general properties that can be discovered or read off from that existing reality.

De Jongh Theorem is an early and striking example of that kind of approach. Many other results of the same type followed, most notably those in provability logic.

Part II: Provability Logic and its topological interpretation

Lindenbaum algebra of a theory T:
$\mathcal{L}_T = \{\text{sentences of } T\}/\sim_T$, where

$$\varphi \sim_T \psi \iff T \vdash (\varphi \leftrightarrow \psi)$$

$\mathcal{L}_T$ is a boolean algebra with operations $\wedge$, $\vee$, $\neg$.
$\mathbf{1}$ = the set of provable sentences of T
$\mathbf{0}$ = the set of refutable sentences of T

For consistent gödelian T all such algebras are countable atomless, hence pairwise isomorphic.

Kripke, Pour-El: even computably isomorphic

Magari algebras

Emerged in 1970s: Macintyre/Simmons, Magari, Smoryński, ...

Consistency operator $\Diamond : \mathcal{L}_T \to \mathcal{L}_T$

$$\varphi \longmapsto \mathsf{Con}(T + \varphi).$$

$(\mathcal{L}_T, \Diamond)$ = *Magari algebra of T*
$\Box\varphi = \neg\Diamond\neg\varphi$ = "φ is provable in T"

Characteristic of $(M, \Diamond)$:
$ch(M) = \min\{k : \Diamond^k \mathbf{1} = \mathbf{0}\}$;
$ch(M) = \infty$, if no such k exists.

Remark. If $\mathbb{N} \models T$, then $ch(\mathcal{L}_T) = \infty$.

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Identities of Magari algebras

K. Gödel (33), M.H. Löb (55): Algebra $(\mathcal{L}_T, \Diamond)$ satisfies the following set of identities *GL*:

- boolean identities
- $\Diamond \mathbf{0} = \mathbf{0}$
- $\Diamond(\varphi \vee \psi) = (\Diamond\varphi \vee \Diamond\psi)$
- $\Diamond\varphi = \Diamond(\varphi \wedge \neg\Diamond\varphi)$ (Löb's identity)

GL-algebras = Magari algebras = diagonalizable algebras

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GL-algebras = Magari algebras = diagonalizable algebras

Provability logic

Let $\mathcal{A} = (A, \Diamond)$ be a boolean algebra with an operator $\Diamond$, and $\varphi(\vec{x})$ a term.

Def. Denote

- $\mathcal{A} \vDash \varphi$ if $\mathcal{A} \vDash \forall \vec{x}\, (\varphi(\vec{x}) = \mathbf{1})$;
- The logic of $\mathcal{A}$ is $Log(\mathcal{A}) = \{\varphi : \mathcal{A} \vDash \varphi\}$.

R. Solovay (76): If $ch(\mathcal{L}_T) = \infty$, then $Log(\mathcal{L}_T, \Diamond) = GL$.

GL is nice as a modal logic (decidable, Kripke complete, fmp, Craig, cut-free calculus, . . .)

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n-consistency

Def. A gödelian theory T is n-consistent, if every provable Σ_n^0-sentence of T is true.

$n\text{-Con}(T) =$ T is n-consistent

n-consistency operator $\langle n \rangle : \mathcal{L}_T \to \mathcal{L}_T$

$$\varphi \longmapsto n\text{-Con}(T + \varphi).$$

$[n] = \neg \langle n \rangle \neg$ (n-provability)

The algebra of n*-provability*

$\mathcal{M}_T = (\mathcal{L}_T; \langle 0 \rangle, \langle 1 \rangle, \dots)$.

The following identities *GLP* hold in $\mathcal{M}_T$:

- *GL*, for all $\langle n \rangle$;
- $\langle n+1 \rangle \varphi \leq \langle n \rangle \varphi$;
- $\langle n \rangle \varphi \leq [n+1] \langle n \rangle \varphi$.

G. Japaridze (86): If $\mathbb{N} \vDash T$, then $Log(\mathcal{M}_T) = GLP$.

GLP is fairly complicated and not so nice modal-logically: no Kripke completeness, no cut-free calculus; though it is decidable and has Craig interpolation.

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Set-theoretic interpretation

Let X be a nonempty set, $\mathcal{P}(X)$ the b.a. of subsets of X.

Consider any operator $\delta : \mathcal{P}(X) \to \mathcal{P}(X)$ and the structure $(\mathcal{P}(X), \delta)$.

Question: Can $(\mathcal{P}(X), \delta)$ be a GL-algebra and, if yes, when?

Def. Write $(X, \delta) \models \varphi$ if $(\mathcal{P}(X), \delta) \models \varphi$. Also let $Log(X, \delta) := Log(\mathcal{P}(X), \delta)$.

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Derived set operators

Let X be a topological space, $A \subseteq X$.
Derived set $d(A)$ of A is the set of limit points of A:

$$x \in d(A) \iff \forall U_x \text{ open } \exists y \neq x \; y \in U_x \cap A.$$

Fact. If $(X, \delta) \models GL$ then X naturally bears a topology τ for which $\delta = d_\tau$, that is, $\delta : A \longmapsto d_\tau(A)$, for each $A \subseteq X$.

In fact, we can define: A is τ-closed iff $\delta(A) \subseteq A$.
Equivalently, $c(A) = A \cup \delta(A)$ is the closure of A.

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Scattered spaces

Definition (Cantor): X is scattered if every nonempty $A \subseteq X$ has an isolated point.

Cantor-Bendixson sequence:

$$X_0 = X, \quad X_{\alpha+1} = d(X_\alpha), \quad X_\lambda = \bigcap_{\alpha<\lambda} X_\alpha, \text{if } \lambda \text{ is limit.}$$

Notice that all X_α are closed and $X_0 \supset X_1 \supset X_2 \supset \dots$

Fact (Cantor): X is scattered $\iff \exists \alpha : X_\alpha = \varnothing$.

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Examples

- Left topology $\tau_{\prec}$ on a strict partial ordering $(X, \prec)$.
  $A \subseteq X$ is open iff $\forall x, y\ (y \prec x \in A \Rightarrow y \in A)$.

  Fact: $(X, \prec)$ is well-founded iff $(X, \tau_{\prec})$ is scattered.

- Ordinal Ω with the usual interval topology generated by intervals (α, β), $[0, \beta)$, (α, Ω) such that $\alpha < \beta$.

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Löb's identity = scatteredness

Simmons 74, Esakia 81

Löb's identity: $\Diamond A = \Diamond(A \wedge \neg\Diamond A)$.

Topological reading:

$$d(A) = d(A \setminus d(A)) = d(\mathrm{iso}(A)),$$

where $\mathrm{iso}(A) = A \setminus d(A)$ is the set of isolated points of A.

Fact: The following are equivalent:

- X is scattered;
- $d(A) = d(\mathrm{iso}(A))$ for any $A \subseteq X$;
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Completeness theorems

Theorem (Esakia 81): There is a scattered X such that $Log(X, d) = GL$. In fact, X is the left topology on a countable well-founded partial ordering.

Theorem (Abashidze/Blass 87/91): Consider $\Omega \geq \omega^\omega$ with the order topology. Then $Log(\Omega, d) = GL$.

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Topological models for GLP

We consider poly-topological spaces $(X; \tau_0, \tau_1, \ldots)$ where modality $\langle n \rangle$ corresponds to the derived set operator d_n w.r.t. τ_n.

Definition: X is a GLP-space if

- τ_0 is scattered;
- For each $A \subseteq X$, $d_n(A)$ is τ_{n+1}-open;
- $\tau_n \subseteq \tau_{n+1}$.

Remark: In a GLP-space, all τ_n are scattered.

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Basic example

Consider a bitopological space (Ω, τ_0, τ_1), where

- Ω is an ordinal;
- τ_0 is the left topology on Ω;
- τ_1 is the interval topology on Ω.

Fact (Esakia): (Ω, τ_0, τ_1) is a model of GLP_2, but not an exact one: linearity axiom holds for $\langle 0 \rangle$, that is,

$$[0](\varphi \to (\psi \vee \langle 0 \rangle \psi)) \vee [0](\psi \to (\varphi \vee \langle 0 \rangle \varphi)).$$

Derivative topology and generated GLP-space

Let (X, τ) be a scattered space.

Fact: There is the coarsest topology τ^+ on X such that $(X; \tau, \tau^+)$ is a GLP_2-space.

The derivative topology τ^+ is generated by τ and $\{d(A) : A \subseteq X\}$ (as a subbase).

Thus, any (X, τ) generates a GLP-space $(X; \tau_0, \tau_1, \ldots)$ with $\tau_0 = \tau$ and $\tau_{n+1} = \tau_n^+$, for each n.

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Completeness for GLP_2

GLP_2 is complete w.r.t. GLP_2-spaces generated from the left topology on a well-founded partial ordering (with Guram Bezhanishvili and Thomas Icard).

Theorem: There is a countable GLP_2-space X such that $Log(X, d_0, d_1) = GLP_2$.

In fact, X has the form $(X; \tau_{\prec}, \tau_{\prec}^{+})$ where $(X, \prec)$ is a well-founded partial ordering.

Aside: This seems to be the first naturally occurring example of a logic that is topologically complete but not Kripke complete.

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Difficulties

Difficulties for three or more operators.

Fact. If (X, τ) is hausdorff and first-countable (i.e. if each point has a countable neighborhood base), then (X, τ^+) is discrete.

Proof: Each $a \in X$ is a unique limit of a countable sequence $A = \{a_n\}$. Hence, $\{a\} = d(A)$ is open.

Ordinal GLP-spaces

Let τ_0 be the left topology on an ordinal Ω. It generates a *GLP*-space $(\Omega; \tau_0, \tau_1, \dots)$. What are these topologies?

Fact: τ_1 is the order topology on Ω.

Club filter topology

Def. Let α be a limit ordinal.

- $C \subseteq \alpha$ is a club in α if C is τ_1-closed and unbounded below α.
- The filter generated by clubs in α is called the club filter. It is improper iff α has countable cofinality.

Fact. τ_2 is the club filter topology:

- τ_2-isolated points are ordinals of countable cofinality;
- if $cf(\alpha) > \omega$ then clubs in α form a neighborhood base of α;
- the least non-isolated point is ω_1.

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Stationary sets

Def. $A \subseteq \alpha$ is stationary in α if A intersects every club in α.

We have: $d_2(A) = \{\alpha : \ cf(\alpha) > \omega$ and $A \cap \alpha$ is stationary$\}$

Remark: Set theorists call d_2 Mahlo operation.
Ordinals in $d_2(Reg)$, where Reg is the class of regular cardinals, are called weakly Mahlo cardinals. Their existence implies consistency of ZFC.

Stationary reflection

Studied by: Solovay, Harrington, Jech, Shelah, Magidor, and many more.

Def. Ordinal κ is *reflecting* if whenever A is stationary in κ there is an $\alpha < \kappa$ such that $A \cap \alpha$ is stationary in α.

Def. Ordinal κ is *doubly reflecting* if whenever A, B are stationary in κ there is an $\alpha < \kappa$ such that both $A \cap \alpha$ and $B \cap \alpha$ are stationary in α.

Theorem. κ is τ_3-nonisolated iff κ is doubly reflecting.

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Corollaries

Fact.

- If κ is weakly compact then κ is doubly reflecting.
- (Magidor) If κ is doubly reflecting then κ is weakly compact in L.

Cor. Assertion "τ_3 is non-discrete" is equiconsistent with the existence of a weakly compact cardinal.

Cor. It is consistent with ZFC that τ_3 is discrete and hence that GLP_3 is incomplete w.r.t. any ordinal space.

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Summary

Let θ_n denote the first limit point of τ_n.

	name	θ_n	$d_n(A)$
τ_0	left	1	$\{\alpha : A \cap \alpha \neq \varnothing\}$
τ_1	order	ω	$\{\alpha \in Lim : A \cap \alpha$ is unbounded in $\alpha\}$
τ_2	club	ω_1	$\{\alpha : cf(\alpha) > \omega$ and $A \cap \alpha$ is stationary in $\alpha\}$
τ_3	Mahlo	θ_3	$\ldots \ \ldots$

θ_3 is the first doubly reflecting cardinal.

On the location of the least non-isolated point

Definition. Let θ_n denote the first non-isolated point of τ_n (in the space of all ordinals).

We have: $\theta_0 = 1$, $\theta_1 = \omega$, $\theta_2 = \omega_1$, $\theta_3 = ?$

ZFC does not know much about the location of θ_3:

- θ_3 is regular, but not a successor of a regular cardinal;
- While weakly compact cardinals are non-isolated, θ_3 need not be weakly compact: If infinitely many supercompact cardinals exist, then there is a model where $\aleph_{\omega+1}$ is doubly reflecting (Magidor);
- If θ_3 is a successor of a singular cardinal, then some very strong large cardinal hypothesis must be consistent (Woodin cardinals).

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- If θ_3 is a successor of a singular cardinal, then some very strong large cardinal hypothesis must be consistent (Woodin cardinals).

On the location of the least non-isolated point

Definition. Let θ_n denote the first non-isolated point of τ_n (in the space of all ordinals).

We have: $\theta_0 = 1$, $\theta_1 = \omega$, $\theta_2 = \omega_1$, $\theta_3 =?$

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Completeness of GLP_2 for Ω

A. Blass (91): 1) If $V = L$ and $\Omega \geq \aleph_\omega$, then GL is complete w.r.t. (Ω, τ_2). (Hence, "GL is complete" is consistent with ZFC.)

2) On the other hand, if there is a weakly Mahlo cardinal, there is a model of ZFC in which GL is incomplete w.r.t. (Ω, τ_2) (for any Ω).

(This is based on a model of Harrington and Shelah in which $\aleph_2$ is reflecting for stationary sets of ordinals of countable cofinality.)

Theorem (B., 2009): If $V = L$ and $\Omega \geq \aleph_\omega$, then GLP_2 is complete w.r.t. $(\Omega; \tau_1, \tau_2)$.

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Further topologies: a conjecture

Theorem (B., Philipp Schlicht): If κ is Π^1_n-indescribable, then κ is non-isolated w.r.t. τ_{n+2}. Hence, if Π^1_n-indescribable cardinals below Ω exist for each n, then all topologies τ_n are non-discrete.

Conjecture: If $V = L$ and Π^1_n-indescribable cardinals below Ω exist for each n, then GLP is complete w.r.t. Ω.

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Topological completeness

GLP is complete w.r.t. (countable, hausdorff) *GLP*-spaces.

Theorem (B., Gabelaia 13): There is a countable hausdorff *GLP*-space X such that $Log(X) = GLP$.

In fact, X is ε_0 equipped with topologies refining the order topology, where $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$.

If *GLP* complete w.r.t. a *GLP*-space X, then all topologies of X have Cantor-Bendixson rank $\geq \varepsilon_0$.

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Conclusions

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- L. Beklemishev, D. Gabelaia, Topological interpretations of provability logic, Leo Esakia on duality in modal and intuitionistic logics, Outstanding Contributions to Logic, 4, eds. G. Bezhanishvili, Springer, 2014, 257290.

Thank you!