136 years and still going strong(?): Cantor's continuum problem

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Dedicated to **Dick de Jongh** and **Anne Troelstra** on the occasion of their 75th birthday

First version: Every infinite $A \subseteq \mathbb{R}$ is either countable or of the same size as $\mathbb{R}$.

The continuum hypothesis, CH

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Second version: The cardinality of $\mathbb{R}$ is $\aleph_1$ (or shorter: $2^{\aleph_0} = \aleph_1$).

Cantor's Set Theory

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- Cantor founded " **Mengenlehre**" (set theory) in the years 1874 to 1897.

  In 1877 it was called **Mannigfaltigkeitslehre**.

- *Unter einer* **‚Menge'** *verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unserer Anschauung oder unseres Denkens (welche die ‚Elemente' von M genannt werden) zu einem Ganzen.* (1895)

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  Cantor's language seems to suggest that 'collection' (Zusammenfassung) is an operation of the mind; in this case the requirement would be that a structural property be represented to the mind according to which the operation of collection is performed.

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'If we want to order the set of these functions in the way required by the problem of the continuum, we must consider how an individual function is **generated***.'*

Initial cases

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Theorem: (Bendixson-Cantor)

If $A \subseteq \mathbb{R}$ is closed, then

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If $A \subseteq \mathbb{R}$ is closed, then

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where P is perfect (closed and has no isolated points), S is countable and $P \cap S = \emptyset$.

Corollary:

Every closed uncountable $A \subseteq \mathbb{R}$ has the same cardinality as $\mathbb{R}$.

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In particular, CH holds for Borel sets.

Adding strong large cardinal assumptions this can be extended to all sets of reals in the **projective hierarchy** obtained from the Borel sets by applying the operations of complement, countable intersection and union, and taking continuous images.

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$$\{Y \subseteq X \mid Y \text{ is definable in } \langle X; \in \rangle \text{ with parameters}\}$$

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Theorem: (Gödel 1938)

L is a model of **ZF** *and of* **AC** *and the generalized continuum hypothesis.*
If **ZF** *is consistent then so is* **ZF** $+$ **AC** $+$ **GCH**.

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Theorem: (Cohen 1963)

If **ZF** *is consistent then so are* $\mathbf{ZF} + \neg CH$ *and* $\mathbf{ZF} + \neg \mathbf{AC}$.

What Hope for Gödel's Program to settle CH?

Theorem (**Cohen; Levy and Solovay** 1967): **CH** is consistent with and independent of all “small" and “large") **LCAs** that have been considered to date, provided they are consistent with **ZF.**

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Proof. By **Cohen**'s method of forcing.

It is consistent for the continuum to be anything not cofinal with ω. This is necessary as by Julius König's Theorem $\mathrm{cf}(2^{\aleph_0}) > \aleph_0$.

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That is, prove

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- Set theory constitutes an ontological foundation for the rest of mathematics.
- There is a unique absolute background concept of set, instantiated in the cumulative universe of all sets, V.
- Set-theoretic questions (e.g. CH) have a definite final answers in V.
- The pervasive independence phenomenon in set theory is due to the weakness of our theories in finding truth, rather than about the truth itself.

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- Set theorists study the models of set theory and how they are connected. They move with agility from one model to another.
- Von Neumann in 1925, in view of Skolem's and Löwenheim's insights, considered the unsettling possibility of one universe of set theory sitting inside another, where properties of sets like "finite" and "well-founded" would shift when moving between universes.

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1. $V[G] \models \neg CH$, collapsing no cardinals, adding no new reals.
2. $V[H] \models CH$, adding no new reals.
3. Thus we have universes

$$V[G_0] \subset V[G_1] \subset \ldots \subset V[G_n] \subset \ldots$$

such that $V[G_{2i}] \models CH$ and $V[G_{2i+1}] \models \neg CH$, having all the same real numbers.

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- $H(\aleph_1)$ is essentially the structure $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, 0, +, \cdot, \in \rangle$ of second order arithmetic.
- While *CH* is not expressible in $H(\aleph_1)$, it's failure is expressible via a Π_2 of the structure $H(\aleph_2)$.

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- Woodin: "*The only known examples of unsolvable problems about the projective sets, in the context of Projective Determinacy, are analogous to the known examples of unsolvable problems in number theory: Gödel sentences and consistency statements."*

Theorem

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2. For each $k \in \mathbb{N}$ there exists a countable transitive set M such that

$$\langle M, \in \rangle \models \textbf{ZFC} + \text{“There exist } k \text{ Woodin cardinals"}$$

and such M is countably iterable.

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- **Theorem.** (Woodin). Suppose that the axiom Martin's Maximum holds. Then there exists a surjection $\rho : \mathbb{R} \to \aleph_2$ such that $\{(x, y) \mid \rho(x) < \rho(y)\}$ is a projective set.

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- Assuming class many Woodin cardinals there is a transfinite hierarchy which extends the hierarchy of the projective sets; this is the hierarchy of the universally Baire sets. Using these sets, Woodin defined a specific strong logic, Ω-logic.

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- Assuming the Strong Ω Conjecture, there are 'good' theories which **maximize** the Π_2-theory of the structure

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- All such theories entail $\neg CH$.
- There is a maximal such theory and in it $2^{\aleph_0} = \aleph_2$ holds.

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- No contradictions have been found (by very smart people).
- Extension principles: “the theory of legitimate candidates"
- Large cardinals exist by analogy with ω (e.g. strongly compact cardinals).

Gödel's Extrinsic Program (1947)

"There might exist axioms so abundant in their verifiable consequences, shedding so much light upon a whole discipline...that quite irrespective of their intrinsic necessity they would have to be assumed in the same sense as any well-established physical theory."

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- “Ultimately of far more significance for this book is that recent results concerning the inner model program undermine the philosophical framework for this entire work."
- “I think the evidence now favors *CH*."
- “The picture that is emerging now [...] is as follows. The solution to the inner model problem for one **supercompact cardinal** yields the ultimate enlargement of *L*. This enlargement of *L* is compatible with all stronger large cardinal axioms and strong forms of covering hold relative to this inner model."

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(a) Generally I do not think that the fact that a statement solves everything really nicely, even deeply, even being the best semi-axiom (if there is such a thing, which I doubt) is a sufficient reason to say it is a “true axiom". In particular I do not find it compelling at all to see it as true.

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(b) The judgments of certain semi-axioms as best is based on the groups of problems you are interested in. For the California school, descriptive set theory problems are central. While I agree that they are important and worth investigating, for me they are not “the center". Other groups of problems suggest different semi-axioms at best; other universes may be the nicest from a different perspective.

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Ultimate goal: If the axiom-candidates following from a given criterion are compatible with set-theoretic practice and, ideally, if there is extrinsic evidence for them, then they are proposed as **new and true axioms** of set theory.

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Current work suggests:
"Small large cardinals" exist.
"Large large cardinals" exist only in inner models.
The Continuum Hypothesis is false.

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Theorem: (Freiling)

(ZFC) $\quad A_{\aleph_0} \Leftrightarrow \neg CH.$

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  So they objected from a perspective of deep experience with non-measurable sets and paradoxical compositions and the role of **AC** therein.

- (Sierpinski) If $\prec$ is a well-ordering of $\mathbb{R}$ of length $\aleph_1$ then the set

$$S := \{(x, y) \mid x \prec y\}$$

  is non-measurable, since it violates the Fubini property:

$$0 = \int_{[0,1]} (\int_{[0,1]} 1_S(x, y) dx) dy = \int_{[0,1]} (\int_{[0,1]} 1_S(x, y) dy) dx = 1$$

The future in 1999

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- Mumford: "*it follows that the C.H. is false and we will get rid of one of the meaningless conundrums of set theory. The continuum hypothesis is surely similar to the scholastic issue of how many angels can stand on the head of a pin: an issue which disappears if you change your point of view.*"

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- Note that PSA is a consequence of GCH.

Applications of CH *I*

- **Emil Artin** conjectured that for all primes p, the p-adic field $\mathbb{Q}_p$ is C_2, i.e., every homogeneous polynomial with $n > d^2$, where n is the number of variables and d is the degree, has a nontrivial zero.

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Theorem: (Ax,Kochen)

For each degree $d \geq 1$, there exists a finite set of primes $P(d)$ such that for all $p \notin P(d)$, if f is a homogeneous polynomial over $\mathbb{Q}_p$ of degree d in n variables such that $n > d^2$, then f has a nontrivial zero in $\mathbb{Q}_p^n$.

Theorem: (Ax-Kochen Principle)

. Any first-order logical statement about valued fields which is true of all but finitely many of the fields $\mathbb{F}_p((t))$ (of formal Laurent series over $\mathbb{F}_p$ is true of all but finitely many of the fields $\mathbb{Q}_p$.

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Uses *CH* to show that if $\mathcal{U}$ is a non-principal ultrafilter on $\mathbb{N}$, then

$$\prod_p \mathbb{F}_p((t))/\mathcal{U} \cong \prod_p \mathbb{Q}_p/\mathcal{U}$$

Applications of CH II

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- Let H be a complex Hilbert space and $\mathcal{B}(H)$ be the set of all bounded linear operators on H. $\mathcal{B}(H)$ is a Banach space with norm

$$\| A \| = \sup\{\frac{\|Av\|}{\|v\|} \mid v \neq 0\}$$

where $\| v \| = \sqrt{\langle v, v \rangle}$, and the product of two operators being defined to be their composition. Additionally there is an involution $* : \mathcal{B}(H) \to \mathcal{B}(H)$ that takes an operator A to its adjoint A^*, which is characterized by the condition

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- An operator $A \in \mathcal{B}(H)$ is of *finite rank* if its range is finite-dimensional. The closure of the set of all operators of finite rank is the set of *compact operators*, $\mathcal{K}(H)$. $\mathcal{K}(H)$ is a C^*-algebra and it is also a closed two-sided ideal of $\mathcal{B}(H)$.

- ℓ^2 is the Hilbert space of infinite sequences $\mathbf{z} = (z_1, z_2, z_3, \ldots)$ of complex numbers z_i such that $\sum_1^\infty |z_n|^2$ converges, equipped with the inner product $\langle \mathbf{z}, \mathbf{w} \rangle = \sum_1^\infty z_n \bar{w}_n$.

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Every *ZFC* model has a forcing extension in which *TA* holds.

- **Weaver** 2006: “Interestingly, it appears that C^*-algebraists generally tend to regard a problem as solved when it has been answered using *CH*.

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This may have to do with the fact that in most cases the other direction of the presumed independence result would involve set theory at a substantially more sophisticated level".

Conservativity of CH

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1. (Shoenfield 1961, Platek 1969)

$$\mathbf{ZF} + \mathbf{AC} + \mathbf{GCH} \vdash \varphi \quad \Rightarrow \quad \mathbf{ZF} \vdash \varphi$$

for $\varphi \in \Pi^1_4$.

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2. (Platek 1969, Silver, Kripke)

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Peter Koellner's Templeton project.

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Solomon Feferman:
Is the continuum hypothesis a definite mathematical problem?

Indefinitely extensible concepts

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Ich setze voraus, dass man wisse, was der Umfang eines Begriffes sei.

I assume that it is known what the extension of a concept is.

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In **Frege: Philosophy of Mathematics**, **Dummett**'s diagnosis of the failure of Frege's logicist project focusses on the adoption of classical quantification. He rejects it in favor of the intuitionistic interpretation of quantification over the relevant domains.

Dummett argues that classical quantification is illegitimate when the domain is given as the objects which fall under an indefinitely extensible concept.

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A is a **definite totality** iff the logical operation of quantifying over A, $\forall x \in A\, P(x)$, has a determinate truth value for each **definite property** $P(x)$ of elements of A.

The Structure of all Sets

V, where V is the universe of all sets, **is not a definite totality**, so unbounded classical quantification over V is not justified on this conception. Indeed, it is essentially indefinite.

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- **Classical logic** for bounded (Δ_0) formulas.
  Heyting's logic for unbounded quantification.

Actualism *versus* Potentialism

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- One way of formally regimenting this informal distinction is by employing intuitionistic logic for domains for which one is a potentialist and reserving classic logic for domains for which one is an actualist.
- This is the approach Tait takes in his work on reflection principles.
  Feferman work on semi-intuitionistic systems of set theory can also be recast in those terms.

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- Quantification over indefinite domains may still be regarded as meaningful, in order to state generic properties, e.g. closure under certain operations performed on sets (e.g. pairing, union etc.).

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- Note that **T** proves full **Replacement** and **Strong Collection** (considered by Tharp, Beeson, Aczel).
- **T** is quite strong. It proves every theorem of (classical) second order arithmetic. In strength it resides strictly between **second order arithmetic** and **Zermelo set theory**.

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- Only $L[A]$ is interesting for our purposes.

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for the result of applying the partial function with code e to $\vec{a}$ (if it exists).

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- For realizers we use codes of Σ_1 partial functions, i.e. Σ_1 definable (with parameters) in the structure $\langle L[A], \in, A \rangle$.
- If e is such a code and $a_1, \ldots, a_n$ are sets in $L[A]$, we use

$$[e]^{L[A]}(a_1, \ldots, a_n)$$

  for the result of applying the partial function with code e to $\vec{a}$ (if it exists).
- In this way the structures $\langle L[A], \in, A \rangle$ give rise to **partial combinatory algebras** (**pca's**) or models of **App**.

Realizability over $\langle L[A], \in, A \rangle$

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\begin{array}{rcl}
e \Vdash a \in b & \text{iff} & a \in b \\
e \Vdash a = b & \text{iff} & a = b \\
e \Vdash \varphi \wedge \psi & \text{iff} & (e)_0 \Vdash \varphi \text{ and } (e)_1 \Vdash \psi \\
e \Vdash \varphi \vee \psi & \text{iff} & [(e)_0 = 0 \wedge (e)_1 \Vdash \varphi] \text{ or } [(e)_0 = 1 \wedge (e)_1 \Vdash \psi] \\
e \Vdash \varphi \to \psi & \text{iff} & \forall d\, [d \Vdash \varphi \Rightarrow [e]^{L[A]}(d) \Vdash \psi] \\
e \Vdash \exists x \theta(x) & \text{iff} & (e)_1 \Vdash \theta((e)_0) \\
e \Vdash \forall x \theta(x) & \text{iff} & \forall a \in L[A]\, [e]^{L[A]}(a) \Vdash \theta(a).
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Above, for a set-theoretic pair $b = \langle u, v\rangle$, we used the notations $(b)_0 = u$ and $(b)_1 = v$. If b is not a pair let $(b)_0 = (b)_1 = 0$.

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Theorem 1. We need a more useful result that exhibits the underlying uniformity. If $\mathcal{D}$ is a **T**-derivation of a formula $\psi(x_1, \ldots, x_n)$, one explicitly constructs a hereditarily finite set $e_{\mathcal{D}}$ such that for all A and all $a_1, \ldots, a_n \in L[A]$,

$$[e_{\mathcal{D}}]^{L[A]}(a_1, \ldots, a_n, \mathbb{R}^{L[A]}) \Vdash \psi(a_1, \ldots, a_n).$$

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Another way of expressing the uniformity and effectiveness of $e_{\mathcal{D}}$ is obtained by viewing $\langle L[A], \in, A\rangle$ as an applicative structure. According to this view, $e_{\mathcal{D}}$ is given by an applicative term t of the theory **App** such that $t\downarrow$ in $L[A]$, i.e.

$$L[A] \models \exists e\, [t \simeq e \wedge e \Vdash \psi].$$

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- We now code the set of reals $\mathbb{R}$ via a set A of ordinals in such a way that the set of real numbers of V_0 belongs to $L[A]$. We thus have
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- Clearly,
$$L[A] \models \neg CH.$$

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- Since $L[A] \models \neg CH$ we must have for $d := [e]^{L[A]}(\mathbb{R}^{L[A]})$ that

$$(d)_0 = 1 \wedge L[A] \models \forall b\, b \nVdash CH.$$

- Since the statement "$[e]^{L[A]}(\mathbb{R}^{L[A]}) \simeq d$" is $\Sigma_1^{L[A]}$, there exists a π such that

$$d, A, \mathbb{R}^{L[A]} \in L_\pi[A] \wedge L_\pi[A] \models [e]^{L_\pi[A]}(\mathbb{R}^{L[A]}) \simeq d.$$

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- **CONTRADICTION!** as $L[A \cup B] \models d \Vdash CH \vee \neg CH$, which implies $(d)_0 = 0$ by (a).

dank u wel

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Gelukkig verjaardag!