Logicist Foundations meet Predicativism
Work in Progress

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Frege's Program

Grundgesetze

Predicative Frege Arithmetic

A Truly Predicative Treatment of Arithmetic?

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Motivation

This result is driven by the hypothesis that there are insights of great interest hidden in traditional approaches that *prima facie* failed. We are in a better position to find these insights because we have new methods (e.g. recursive saturation) and ideas (constructivism, Martin-Löf Type Theory) available.

Well ... to be honest ... it is also simply fun to think about these things.

What is this Talk About

1. I explain Frege's abstractionist logicism. I will suggest that it is *incomplete*. One way of completing it, is to put it on a predicativistic basis.
2. I introduce the usual predicative versions of Abstractionism.
3. I explain why these versions are are not truly predicative.
4. I handwave at what I think a truly predicative development should look like. These ideas have to be tested.
5. Possible problems with my approach.

The Story

1847 Boole: The Mathematical Analysis of Logic

1879 Frege: Begriffsschrift

1884 Frege: Grundlagen der Arithmetik

1888 Dedekind: Was sind und was sollen die Zahlen

1889 Peano: The principles of arithmetic, presented by a new method

1893 Frege: Grundgesetze der Arithmetik 1

1895-1997 Cantor: Beiträge zur Begründung der transfiniten Mengenlehre

1902 Russell: the paradox

1903 Frege: Grundgesetze der Arithmetik 2

1908 Ernst Zermelo: Investigations in the foundations of set theory I

1910, 1912, 1913 Russell & Whitehead: Principia Mathematica

Logicism

Logicism is the attempt to found number theory and, more generally, (a substantial portion of) mathematics on the Basic Laws of Thought. This in oppostion to Kant who thought that arithmetic is *synthetic a priori*.

We focus on Frege's brand of Logicism
Distinguishing features:

- Special Notation
- Abstractionism (in Grundlagen) We focus on this one.
- Centrality of function-application

Frege

Figure: Gottlob Frege

Overview

Basic Setting

We work in many-sorted logic. We have a sort of basic objects or urelements, a sort of extensions, a sort of concepts / classes, a sort of binary concepts / binary relations. To simplify a bit we will ignore the sort of objects.

We have the usual formula classes Π^1_n, Σ^1_n and Δ^1_n. $\Delta^1_0 = \Pi^1_0 = \Sigma^1_0$ consists of all formulas without concept quantifiers. Δ^1_0 is also called the class of *predicative* formulas.
Warning: we will not generally have pairing (of sufficiently low complexity), so e.g. Π^1_1 is of the form $\forall X_0 \ldots \forall X_{n-1}\, \phi$, where ϕ is in Δ^1_0.

As long as we have Π^1_1-comprehension and Law V we can simulate binary concepts since we will have pairing on the ground domain, but in general their presence will make a difference.

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Abstraction: Grundlagen

Suppose E is an equivalence relation on D then we have a function $@_E$ and a domain A_E such that:

- $@_E d = @_E e$ iff $d\ E\ e$.

Typical examples are the introduction of *directions* and *distances*.

There is a fundamental problem with the individuation of the abstracts thus introduced called the *Julius Caesar Problem*. *How do we know that the number 5 is not Julius Caesar?*

Overview

The System

The system GG(Γ) is defined as follows:

- Γ-comprehension.
- Extensionality of concepts. As long as we have no higher order concepts this is cheap.
- Law V: $\partial X = \partial Y$ iff $X = Y$.

The Russell Paradox

We define $x \in y :\leftrightarrow \exists Y\, (\partial(Y) = y \wedge Y(x))$. Suppose R is the concept of not-being-in-oneself, i.e. the concept of being an x such that $x \notin x$. Let $\partial(R) = r$.

Suppose $r \in r$. Then, for some Y, $\partial(Y) = r$ and $Y(r)$. It follows that R is co-extensional with Y, and, thus, $R(r)$, and, hence $r \notin r$. This contradicts our assumption that $r \in r$.

We may conclude that $r \notin r$, and, thus, $R(r)$. It follows that $r \in r$. An outright contradiction.

The theory $\mathrm{GG}(\Delta^1_1)$ is consistent (Ferreira-Wehmeier, Walsh), but Π^1_1-comprehension is inconsistent. $\mathrm{GG}^-(\Delta^1_1)$ *proves* that there are non-extensions.

Predicativism and GG(Δ^1_0)

Predicativism embodies a kind of non-circularity and grounding condition. A concept can only be introduced (in abstract conceptual time) when is objects already exist. The definition of a concept may not appeal to objects or concepts not yet created and specifically not to the concept under consideration itself.

Δ^1_0-comprehension is the usual predicative system. We can see that quantification over the totality of concepts —which would include the concept that is being defined— is forbidden in the definition of a concept.

There is an opposition with the Platonic view of definition where with definitions we only select something already existing. (This entangles us in knotty semantical problems, since conceptual time or the conceptual order is not real time . . . .)

Strength of GG(Δ^1_0)

Even if, as we will suggest, the philosophical credentials of Δ^1_0-comprehension are suspect, it is a very meaningful expansion of a theory from a metamathematical point of view because of the connection with consistency.

GG(Δ^1_0) is mutually interpretable with Q (Ganea, Visser). (This result is very robust for variations of detail.)

Repeating the construction gives us a hierarchy that corresponds to iterating consistency statements over Q (Visser). It also follows a hierarchy of functions defined by Alex Wilkie (Visser, unpublished). $\mathrm{I}\Delta_0 +$ supexp is the unattainable upperbound for the hierarchy.

Models of $GG(\Delta^1_0)$ part 1

We assume that for each n we have n-ary concepts. Let M be any first-order structure. The class $\mathrm{Def}(M^n) \subseteq P(M^n)$ consists of the $X \subseteq M^n$ definable with parameters in M.

If M is infinite, $\mathrm{Def}(M)$ and M have the same cardinality. Hence, we can choose $\partial : \mathrm{Def}(M) \to M$ to be any injection. Then the following structure is a model of $GG(\Delta^1_0)$:

$$\mathcal{M} = (M, \mathrm{Def}(M), \mathrm{Def}(M \times M), \ldots, \partial)$$

Injectivity implies Basic Law V. The verification of predicative comprehension is on next slide.

Models of GG(Δ^1_0) part 2

Consider predicative comprehension for A.

$$\vdash \exists X^n \forall \vec{x}\,(\,\vec{x} \in X^n \leftrightarrow A(\vec{x}, \vec{y}, \vec{Z})\,). \tag{1}$$

A class variable Z has three kinds of occurrences in A:

(a) in a formula of the form $Z = U$ or $U = Z$,

(b) in a formula of the form $t = \partial Z$ or $\partial Z = t$ or $(\ldots, \partial Z, \ldots) \in U$,

(c) in a formula of the form $\vec{t} \in Z$.

Eliminate subformulas (a) by replacing them by $\forall \vec{u}\,(\vec{u} \in Z \leftrightarrow \vec{u} \in U)$. In subformulas (b) we replace ∂P by the value p of ∂P in the model. We replace in subformulas (c) $\vec{t} \in P$ by $B(\vec{t}, \vec{d}', \vec{P}')$, where B is the first-order definition of P. We take the final formula as the definition of X^n.

Simple Frege Models

A *simple Frege model* is a structure of the following form

$$\mathcal{M} = (\omega, \mathsf{Def}(\omega), \mathsf{Def}(\omega \times \omega), \ldots, \partial)$$

where on ω we have the language of equality.

The definable subsets of ω are precisely the finite and the cofinite sets.

Many Non-Isomorphic Simple Frege Structures

There are many non-isomorphic simple Frege models.

Let $(X_i)_{i \in \omega}$ be an enumeration without repetitions of the definable classes of ω except the singletons. We define $\partial_0(X_i) := 2i$ and $\partial_0(\{k\}) := 2k + 1$. So according to ∂_0 no object codes its own singleton.

Let $(Y_i)_{i \in \omega}$ be an enumeration without repetitions of the definable classes of ω except the singletons of odd numbers. We define $\partial_1(Y_i) := 2i$ if X_i and $\partial_1(\{2k + 1\}) := 2k + 1$. So according to ∂_1 there are infinitely many elements that code their own singleton.

This basic non-categoricity is even a problem if ∂ is surjective. Frege's approach is radically underdetermined.

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The System

$\text{HP}^2(\Gamma)$ is defined like $\text{GG}(\Gamma)$ with Law V replaced by:

- Hume's Principle: $\#X = \#Y$ iff there is a bijection $F : X \to Y$ (if X and Y are *equinumerous*).

Full HP is paradox-free!

One can show that Predicative V and Predicative Frege Arithmetic are mutually interpretable but as far as I know there is no quick and easy argument for any of the two directions.

The Linnebo model

The Linnebo model $\mathbb{L}$ has as basic domain $\widetilde{\omega}$ which is defined by: $0, 1, 2, \ldots \; \ldots, \infty - 2, \infty - 1, \infty$. We just have identity on $\widetilde{\omega}$.

The classes are the (parametrically) definable classes in the language of identity over $\widetilde{\omega}$. In the unary case these are the finite and cofinite subsets of $\widetilde{\omega}$.

We define $\#X := n$ if X has n elements (for n a natural number) and $\#X := \infty - m$ if $\widetilde{\omega} \setminus X$ has m elements, for m a natural number.

It seems that HP is doing better than V w.r.t. the uniqueness of models. Some further study is needed of the question: *How categorical (schmategorical) is* HP*?*

Some Definitions

- $0 := \#\emptyset$.
- $\infty := \#\mathrm{V}$,
  where $\mathrm{V} = \{x \mid x = x\}$.
- $p \leq q :\leftrightarrow \exists X, Y\,(\#X = p \wedge \#Y = q \wedge X \subseteq Y)$.
- $p < q :\leftrightarrow \exists X, Y\,(\#X = p \wedge \#Y = q \wedge X \subset Y)$.
- $\mathrm{S}(p, q) :\leftrightarrow \exists X, x\,(x \notin X \wedge \#X = p \wedge \#(X \cup \{x\}) = q)$.
- A cardinal p is *Dedekind finite* iff $p \not< p$.
- $\mathrm{bebu}(n) :\leftrightarrow n = \#\{p \mid p < n\} \wedge n \not< n$.
- $\mathrm{her}(X) :\leftrightarrow 0 \in X \wedge \forall p, q\,((\mathrm{S}(p, q) \wedge p \in X) \rightarrow q \in X)$.
- $\mathrm{freg}(n) :\leftrightarrow \forall X\,(\mathrm{her}(X) \rightarrow n \in X)$.

Yes We Can and No We Cannot

We can verify many of the usual properties of arithmetic in $\mathrm{HP}^2(\Delta^1_0)$. *Yes*, S is an injective partial function and 0 is not in its range. *Yes*, $<$ is a partial pre-ordering with minimum 0 and maximum ∞. The virtual class bebu is provably closed under successor (Beth, Burgess). A variant of bebu even satisfies the induction axiom.

No, we cannot prove that ∞ has a successor. So we cannot exclude that ∞ is Dedekind finite. *No*, we cannot prove that the virtual class freg of Frege's natural numbers is closed under successor. In fact in $\mathbb{L}$ the natural numbers à la Frege are all the cardinals. In $\mathbb{L}$ bebu gives us the standard natural numbers.

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Why Predicative V and Predicative Hume's Principle are *not* Predicative

If we look at the models we see that $\mathrm{GG}(\Delta^1_0)$ is highly non-categorical. We can get some kind of unicity of models by imposing extra conditions but *some extra story is needed to explain why such conditions should obtain*.

Also the basic philosophical idea that a set is determined by its graph is overboard, since for no model of predicative GG, its graph can be bisimulation minimal. If $\mathfrak{v} := \partial\mathrm{V}$ and $\mathfrak{v}^* := \partial(\mathrm{V} \setminus \{\mathfrak{v}\})$, then $\mathfrak{v}$ and $\mathfrak{v}^*$ are bisimilar.

The main problem is that we have unbounded quantification over extensions or numbers. The Δ^1_0-comprehension principle does conform to the idea that the totality of concepts might be open-ended and growing, but it assumes that the extensions / numbers are pre-given. In a defining formula of X we can have a quantification over all numbers / extensions, where presumably ∂X and $\# X$ depend on X for their existence.

Possible Axioms

$\mathrm{hp}^{\mathrm{p}}(1)$ $\vdash x \notin \emptyset$

$\mathrm{hp}^{\mathrm{p}}(2)$ $\vdash \mathrm{id}(\emptyset) \equiv \emptyset \times \emptyset$

$\mathrm{hp}^{\mathrm{p}}(3)$ $\vdash Rxy \rightarrow (x \in \mathrm{dom}(R) \wedge y \in \mathrm{cod}(R))$

$\mathrm{hp}^{\mathrm{p}}(4)$ $\vdash \mathrm{dom}(\mathrm{id}(X)) \equiv X$

$\mathrm{hp}^{\mathrm{p}}(5)$ $\vdash \mathrm{cod}(\mathrm{id}(X)) \equiv X$

$\mathrm{hp}^{\mathrm{p}}(6)$ $\vdash \forall x \in X\ x =_X x$ We write $=_X$ for $\mathrm{id}(X)$.

$\mathrm{hp}^{\mathrm{p}}(7)$ $\vdash \forall x, x' \in X\ (x =_X x' \rightarrow x' =_X x)$

$\mathrm{hp}^{\mathrm{p}}(8)$ $\vdash \forall x, x', x'' \in X\ ((x =_X x' \wedge x' =_X x'') \rightarrow x =_X x'')$

$\mathrm{hp}^{\mathrm{p}}(9)$ $\vdash \forall x, x' \in X\ (x =_X x' \leftrightarrow x = x')$
Note that $=$ is a defined relation. We need to check its properties.

$\mathrm{hp}^{\mathrm{p}}(10)$ $\vdash \#X \equiv \#Y \rightarrow X \equiv Y$

$\mathrm{hp}^{\mathrm{p}}(11)$ $\vdash u \in \{x\} \leftrightarrow u \equiv x$

Possible Axioms 2

$\mathrm{hp^p}(12)\ \vdash \mathrm{id}(\{x\}) \equiv \{x\} \times \{x\}$

$\mathrm{hp^p}(13)\ \vdash (x, y) \in X \times Y \leftrightarrow (x \in X \wedge y \in Y)$

$\mathrm{hp^p}(14)\ \vdash \mathrm{dom}(X \times Y) \equiv X$

$\mathrm{hp^p}(15)\ \vdash \mathrm{cod}(X \times Y) \equiv Y$

$\mathrm{hp^p}(16)\ \vdash x \in \{u \in X \mid \phi u\} \leftrightarrow (x \in X \wedge \phi x)$

$\mathrm{hp^p}(17)\ \vdash \mathrm{id}(\{u \in X \mid \phi(u)\}) \equiv \{(v, w) \in X \times X \mid \phi(u) \wedge \phi(v) \wedge u =_X v\}$

$\mathrm{hp^p}(18)\ \vdash (x, y) \in \{(u, v) \in X \times Y \mid \phi uv\} \leftrightarrow ((x, y) \in X \times Y \wedge \phi xy)$

$\mathrm{hp^p}(19)\ \vdash \mathrm{dom}(\{(u, v) \in X \times Y \mid \phi uv\}) = X$

$\mathrm{hp^p}(20)\ \vdash \mathrm{cod}(\{(u, v) \in X \times Y \mid \phi uv\}) = Y$

Possible Axioms 3

We write: iwi(X, E, Y) for:
$\mathrm{dom}(E) = X \wedge \mathrm{dom}(E) = Y \wedge \forall x \in X\ y \in Y\,(x\,E\,y \leftrightarrow x = y)$.

hp$^{\mathrm{p}}$(21) $\vdash \mathrm{iwi}(X, E, Y) \to X \cup_E Y \downarrow$
hp$^{\mathrm{p}}$(22) $\vdash \mathrm{iwi}(X, E, Y) \to \forall z\,(z \in X \cup_E Y \leftrightarrow (z \in X \vee z \in Y))$
hp$^{\mathrm{p}}$(23) $\vdash \mathrm{id}(X \cup_E Y)(u, v) \leftrightarrow (u =_X v \vee u =_Y v \vee u\,E\,v \vee v\,E\,u)$

A preliminary investigation suggests that (an appropriate version of) bebu is provably closed under successor.

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Ruminations

I think the idea that objects in order to be counted should not merge with each other in conceptual time is not yet sufficiently represented in the system. Go to intuitionistic logic? Work in progress.

A doubt arises: if we have the idea of abstract time anyway, why not go for a Kantian-Brouwerian treatment of number based on the intuition of time? My answer would be: because the last does not deliver the notion of cardinal. For me the whole thing is about the study of the cardinal as a *sui generis* entity.

Another doubt: why not make the tokens / intensions members of the club of countibilia? Since $X \prec \#X$, we have $\#X \notin X$ and so infinity would be for free. (Note that we could have $\exists x\,(\#X = x \wedge x \in X)$), so the insight is strictly a token insight. No idea how to answer this one or whether it should be answered.

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Thank You

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