The Limits of Causal Graphs: Density

Dean McHugh

Institute of Logic, Language and Computation
University of Amsterdam

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Institute for Logic, Language and Computation

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University of Amsterdam

Plan

1. Primer on structural causal models
2. Directness and density
3. Dense causal chains
4. Dense dependence in dynamical systems

Two questions

1. **The modeling question**
   What kind of information do we use when we judge that a causal relation holds?

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Today: the modeling question

What mathematical form should models of causation take?

Structural causal models

Definition (Structural causal model)

A structural causal model is a triple $M = (V, E, F)$ where

- V is a set of variables
- (V, E) is a directed acyclic graph
- F is a set of functions of the form

$$F_X : \mathcal{R}(pa_X) \to \mathcal{R}(X),$$

  one for each endogenous variable $X \in V$,
  where $pa_X := \{Y \in V : (Y, X) \in E\}$

Structural causal models

The value of an endogenous variable X is determined by the values of its parents, according to F_X

- Since F_X are functions, the dependence is deterministic
- Where $U = u$ is an assignment of values to the exogenous variables in V, we call u a *setting* or *context* for M
    - i.e. the values of the exogenous variables determine the values of all the variables

Interventions in structural causal models

Let $M = (V, E, F)$ be a structural causal model

Definition (Interventions as model surgery)

$M_{do(X=x)}$ is the model $(V_{do(X=x)}, E_{do(X=x)}, F_{X=x})$ where

- $V_{do(X=x)} = V$
- $E_{do(X=x)} = E \setminus \{(Y, X) : Y \in V\}$
- $F_{do(X=x),X}(u) = x$ for every setting u of M,
  and $F_{do(X=x),Y} = F_Y$ for all $Y \in V$, $Y \neq X$

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Definition (Truth conditions for interventions)

Let M be a structural causal model and u a setting of the exogenous variables.

$$M, u \models [X \leftarrow x]Y = y \quad \text{iff} \quad M_{do(X=x)}, u \models Y = y$$

SCMs represent dependence as direct

> *In causal diagrams, an arrow represents a "direct effect" of the parent on the child, although this effect is direct only relative to a certain level of abstraction, in that the graph omits any variables that might mediate the effect.*
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In the beginning, dependence was direct

CORRELATION AND CAUSATION

By SEWALL WRIGHT

Senior Animal Husbandman in Animal Genetics, Bureau of Animal Industry, United States Department of Agriculture

PART I. METHOD OF PATH COEFFICIENTS

INTRODUCTION

The ideal method of science is the study of the direct influence of one condition on another in experiments in which all other possible causes of variation are eliminated. Unfortunately, causes of variation often seem to be beyond control. In the biological sciences, especially, one

Figure: Wright (1921)

Dense causal chains

In daily life and physicists' models alike,
space and time are represented as *dense*

Density Between any two points there is a third

Intuitive belief: Causal influence travels through dense spacetime

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There are chains of events $(C_i)_{i \in I}$ where,
for every C_i, C_k on the chain, there is a C_j on the chain such that

$$C_i \text{ causally influenced } C_j \quad \text{and} \quad C_j \text{ causally influenced } C_k.$$

Is density compatible with direct dependence?

Question Can dense dependence be represented in terms of direct dependence?

- Intuitively, dense dependence is indirect

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Is density compatible with direct dependence?

Question Can dense dependence be represented in terms of direct dependence?

- Intuitively, dense dependence is indirect
- One can add more arrows
- But still, more arrows just means more instances of direct dependence

Causation, Coherence, and Concepts pp 99-111 | Cite as

Bayesian Nets Are All There Is to Causal Dependence

Chapter

Part of the Boston Studies in the Philosophy of Science book series (BSPS, volume 256)

Wolfgang Spohn (2009)

A simple dense causal chain

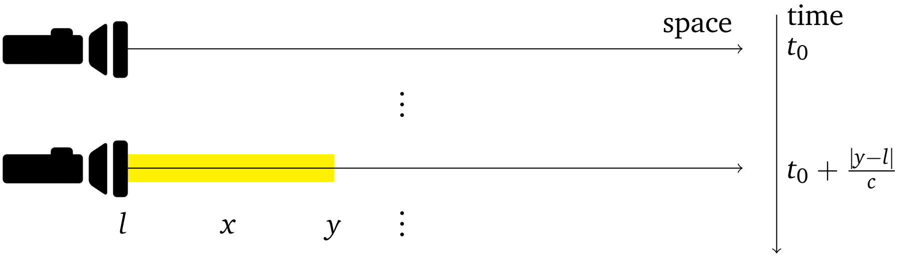

Figure: Turning on the light at time t_0

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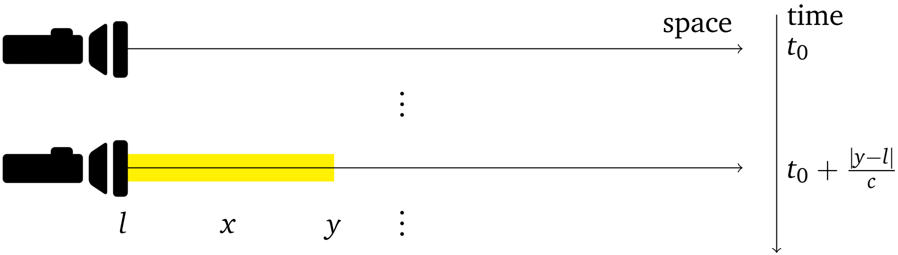

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How do structural causal models represent dependence here?

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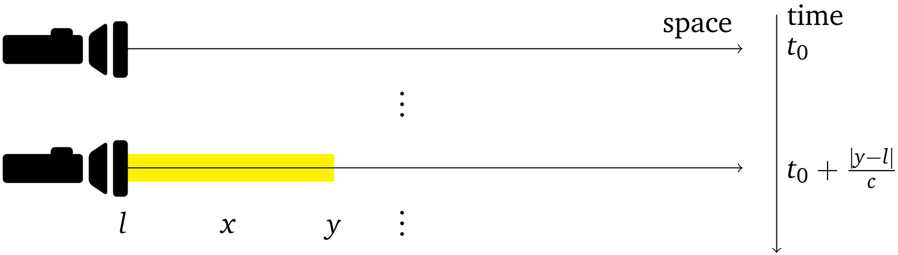

Figure: Turning on the light at time t_0

How do structural causal models represent dependence here?
Answer: A version of counterfactual dependence

(Paul, 1998; Yablo, 2002; Halpern and Pearl, 2005; Halpern, 2016; Beckers, 2016)

- Let x be a point illuminated at t, and $t' = t + \frac{|y-x|}{c}$
- If x had not been illuminated at t, y would not be illuminated at t'

Definition (Functional dependence)

Let Y be a variable, pa_Y a set of variables, and $\vec{X} \subseteq pa_Y$. Let $f_Y : \mathcal{R}(pa_Y) \to \mathcal{R}(Y)$ a function from the values of the variables in pa_Y to the values of Y. We say Y functionally depends on $\vec{X}$ just in case

$$f_Y(\vec{x}, \vec{o}) \neq f_Y(\vec{x}', \vec{o})$$

for some values $\vec{x}, \vec{x}'$ of $\vec{X}$ and value $\vec{o}$ of the variables in $pa_Y - \vec{X}$.

Definition (Dense dependence)

Let Y be a variable and $f_Y : \mathcal{R}(pa_Y) \to \mathcal{R}(Y)$. We say dependence is dense at Y iff for any parents $\vec{X}$ of Y there are parents $\vec{Z}$ of Y such that

$$f_Y(\vec{x}, \vec{z}, \vec{o}) = f_Y(\vec{x}', \vec{z}, \vec{o})$$

for all values $\vec{x}, \vec{x}'$ of $\vec{X}$, value $\vec{z}$ of $\vec{Z}$, and values $\vec{o}$ of $pa_Y - (\vec{X} \cup \vec{Z})$.

Theorem

No structural causal model $M = (V, E, F)$ contains a variable Y such that

1. *Y functionally depends on some variables $\vec{X} \subseteq V$ in M, and*
2. *Dependence is dense at Y in M.*

Proof.

Let $M = (V, E, F)$ be a structural causal model with $Y \in V$ satisfying (1) and (2). By (1), for some values $\vec{x}, \vec{x}'$ of $\vec{X}$ and value $\vec{o}$ of $pa_Y - \vec{X}$,

$$f_Y(x, o) \neq f_Y(x', o).$$

And by (2), for some parents $\vec{Z}$ of Y and value $\vec{z}$ of $\vec{Z}$ we have

$$f_Y(\vec{x}, \vec{z}, \vec{o} - \vec{z}) = f_Y(\vec{x}', \vec{z}, \vec{o} - \vec{z}).$$

But then we have the following contradiction:

$$f_Y(\vec{x}, \vec{o}) \overset{(1)}{\neq} f_Y(\vec{x}', \vec{o}) = f_Y(\vec{x}', \vec{z}, \vec{o} - \vec{z}) \overset{(2)}{=} f_Y(\vec{x}, \vec{z}, \vec{o} - \vec{z}) = f_Y(\vec{x}, \vec{o}).$$

Applying the impossibility result to the light example

- Suppose that variable (i.e. spacetime point) changes its value in response to intervening on the torch.

Applying the impossibility result to the light example

- Suppose that variable (i.e. spacetime point) changes its value in response to intervening on the torch.
  - Then there is variable $X_{t'}$ such that L_t is a parent of $X_{t'}$ and

    $$f_{X_{t'}}(l_t, \vec{o}) \neq f_{X_{t'}}(l'_t, \vec{o})$$

    for some values l_t, l'_t of L_t and value o of $pa_X - \{L_t\}$.

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- Also suppose there is no first spacetime point to the right of the torch.

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- Also suppose there is no first spacetime point to the right of the torch.
- Then there is some point (or region) $\vec{Z}_{t''}$ between the torch and X such that, fixing the value of $\vec{Z}_{t''}$ also fixes the value of X_t.

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- Then there is some point (or region) $\vec{Z}_{t''}$ between the torch and X such that, fixing the value of $\vec{Z}_{t''}$ also fixes the value of X_t.
  - That is, there is set of parents $\vec{Z}_{t''}$ of $X_{t'}$ such that

    $$f_{X_{t'}}(l_t, \vec{z}_{t''}, \vec{o}') = f_{X_{t'}}(l'_t, \vec{z}_{t''}, \vec{o}')$$

    for all values l_t, l'_t of L_t, value $\vec{z}_{t''}$ of $\vec{Z}_{t''}$ and value $\vec{o}'$ of the parents of $X_{t'}$ other than $X_{t'}$ and $\vec{Z}_{t''}$.

Applying the impossibility result to the light example

- Suppose that variable (i.e. spacetime point) changes its value in response to intervening on the torch.
  - Then there is variable $X_{t'}$ such that L_t is a parent of $X_{t'}$ and

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    for some values l_t, l'_t of L_t and value o of $pa_X - \{L_t\}$.
- Also suppose there is no first spacetime point to the right of the torch.
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    for all values l_t, l'_t of L_t, value $\vec{z}_{t''}$ of $\vec{Z}_{t''}$ and value $\vec{o}'$ of the parents of $X_{t'}$ other than $X_{t'}$ and $\vec{Z}_{t''}$.
- But then $f_{X_{t'}}(l_t, \vec{o}) = f_{X_{t'}}(l_t, \vec{z}_{t''}, \vec{o}') = f_{X_{t'}}(l'_t, \vec{z}_{t''}, \vec{o}') = f_{X_{t'}}(l'_t, \vec{o}) \neq f_{X_{t'}}(l_t, \vec{o})$.

Model constraints

A *model* $M = (V, P)$ is a pair consisting of a variable set V and a set of paths P, satisfying some constraints.

$$X_t = Y_{t+\frac{|x-y|}{c}}$$

for all points x, y in space and time t, where c is the speed of light.

Interventions in dynamical systems

We can think of of intervening as gluing two paths together. Given a state of the system at a point in time t, i.e. a valuation assigning to each variable Y_t a value y in the range of Y_t, let

$$do(\vec{X} = \vec{x}, Y_t) = \begin{cases} x & \text{if } Y \text{ is in } \vec{X} \\ y & \text{otherwise} \end{cases}$$

where if Y is in $\vec{X}$, x is the value Y receives in $\vec{X}$.

Definition (The effects of intervening at time t)

$$M_{do(\vec{X}_t = \vec{x})}, u \models Y_{t'} = y \quad \text{iff} \quad \begin{array}{l} t' < t \text{ and } M, u \models Y_{t'} = y, \text{ or} \\ t' = t \text{ and } M, u \models do(\vec{X}_t = \vec{x}, Y_{t'} = y), \text{ or} \\ t' > t \text{ and } M, u' \models Y_{t'} = y \text{ for every } u' \\ \quad \text{with } M, u' \models Z_t = z \text{ iff } M, u \models Z_t = z \\ \quad \text{for every variable } Z_t \end{array}$$

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