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Generalized Inquisitive Semantics and Logic

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Abstract This paper investigates a generalized version of *inquisitive semantics*. A complete axiomatization of the associated logic is established, the connection with intuitionistic logic and several intermediate logics is explored, and the generalized version of inquisitive semantics is argued to have certain advantages over the system that was originally proposed by Groenendijk (2008c) and Mascarenhas (2008).

Keywords inquisitive semantics · inquisitive logic · intuitionistic logic · intermediate logics · Medvedev logic

Contents

1 Introduction

Traditionally, logic is concerned with argumentation. As a consequence, formal investigations of the semantics of natural language are usually focussed on the *descriptive* use of language, and the meaning of a sentence is identified with its *informative* content. Stalnaker (1978) gave this informative notion a dynamic and conversational twist by taking the meaning of a sentence to be its potential to enhance the common ground,

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where the common ground is viewed as the conversational participants' shared information. Technically, the common ground is taken to be a set of possible worlds, and a sentence provides information by eliminating some of these possible worlds.

Of course, this picture is limited in several ways. First, it only applies to sentences that are used exclusively to provide information. Even in a typical informative dialogue, utterances may serve different purposes as well. Second, the given picture does not take into account that updating the common ground is a *cooperative* process. One speech participant cannot simply change the common ground all by herself. All she can do is *propose* a certain change. Other speech participants may react to such a proposal in several ways. These reactions play a crucial role in the dynamics of conversation.

In order to overcome these limitations, *inquisitive semantics* starts with a different picture. It views propositions as proposals to change the common ground. These proposals do not always specify just one way of changing the common ground. They may suggest alternative ways of doing so, among which the responder is then invited to choose. Formally, a proposition consists of one or more *possibilities*. Each possibility is a set of possible worlds and embodies a possible way to enhance the common ground. If a proposition consists of two or more possibilities, it is *inquisitive*: it invites the other participants to respond in a way that will lead to a cooperative choice between the proposed alternatives. Inquisitive propositions raise an issue. They give direction to a dialogue. Thus, inquisitive semantics directly reflects that a primary use of language lies in the exchange of information in a cooperative dynamic process of raising and resolving issues.

Groenendijk (2008c) and Mascarenhas (2008) first defined an inquisitive semantics for the language of propositional logic, focussing on the philosophical and linguistic motivation for the framework, and delineating some of its basic logical properties. The associated logic was axiomatized by Mascarenhas (2009), while a sound and complete sequent calculus was established independently by Sano (2008). Several linguistic applications of the framework are discussed by Balogh (2009).

In this paper, we consider a generalized version of the semantics proposed by Groenendijk (2008c) and Mascarenhas (2008). This generalized semantics was first considered by Groenendijk (2008a). Initially, it was thought to give essentially the same results as the original semantics. Upon closer examination, however, Mascarenhas, Groenendijk, and Ciardelli observed that the two systems are different, and Ciardelli (2008) first argued that these differences speak in favor of the generalized semantics. Groenendijk and Roelofsen (2009) adopted the generalized semantics, and developed a formal pragmatic theory based on it.

The aim of the present paper is threefold. First, we will investigate and present some of the key features of the generalized semantics in a systematic way. Second, we will analyse the logic that the semantics gives rise to. In particular, we will explore the connection with intuitionistic logic and several well-known intermediate logics, which, among other things, will lead to a sound and complete axiomatization of inquisitive logic. Finally, we will argue that the generalized semantics is better-behaved than the original version of inquisitive semantics. In fact, we will define an entire hierarchy of parameterized versions of inquisitive semantics, and argue that only the generalized version, which can be seen as the limit case of the hierarchy, really behaves satisfactorily.

The paper is organized as follows. Section 2 introduces the generalized version of inquisitive semantics and presents some key features of the system. Section 3 investigates the associated logic, leading up to a sound and complete axiomatization. Section 4 shows that the *schematic fragment* of inquisitive logic (the logic itself is not closed under

uniform substitution) coincides with the well-known Medvedev logic of finite problems. This is particularly interesting as it yields a sort of finitary pseudo-axiomatization of Medvedev logic (which is known not to be finitely axiomatizable). Section 6 presents a translation of inquisitive logic into intuitionistic logic, showing that the former can be identified with the disjunctive-negative fragment of the latter. Section 7 defines and axiomatizes an infinite hierarchy of inquisitive logics, one of which is associated with the original semantics, and section 8 argues that the generalized semantics has certain advantages over all the other elements of the hierarchy. Finally, section 9 suggests a new intuitive interpretation of the notion of *support*—a notion that will play a key role in the definition of inquisitive semantics. This intuitive interpretation is intended to illuminate some of the technical results obtained in earlier sections.

2 Generalized Inquisitive Semantics

We assume a language $\mathcal{L}_{\mathcal{P}}$, whose expressions are built up from $\bot$ and a (finite or countably infinite) set of proposition letters $\mathcal{P}$, using binary connectives $\wedge, \vee$ and $\rightarrow$. We will also make use of three abbreviations: $\neg\varphi$ for $\varphi \rightarrow \bot$, $!\varphi$ for $\neg\neg\varphi$, and $?\varphi$ for $\varphi \vee \neg\varphi$. The first is standard, the second and the third will become clear shortly.

2.1 Indices, States, and Support

The basic ingredients for the semantics are *indices* and *states*.

Definition 2.1 (Indices)**.** A $\mathcal{P}-$*index* is a subset of $\mathcal{P}$. The set of all indices, $\wp(\mathcal{P})$, will be denoted by $\mathcal{I}_{\mathcal{P}}$. We will simply write $\mathcal{I}$ and talk of *indices* in case $\mathcal{P}$ is clear from the context.

Definition 2.2 (States)**.** A $\mathcal{P}-$*state* is a set of $\mathcal{P}-$indices. The set of all states, $\wp\wp(\mathcal{P})$, will be denoted by $\mathcal{S}_{\mathcal{P}}$. Again, reference to $\mathcal{P}$ will be dropped whenever possible.

The meaning of a sentence will be defined in terms of the notion of *support* (just as, in a classical setting, the meaning of a sentence is usually defined in terms of truth). Support is a relation between states and formulas. We write $s \models \varphi$ for 's supports φ'.

Definition 2.3 (Support)**.**

1. $s \models p$ iff $\forall w \in s : p \in w$
2. $s \models \bot$ iff $s = \emptyset$
3. $s \models \varphi \wedge \psi$ iff $s \models \varphi$ and $s \models \psi$
4. $s \models \varphi \vee \psi$ iff $s \models \varphi$ or $s \models \psi$
5. $s \models \varphi \rightarrow \psi$ iff $\forall t \subseteq s$: if $t \models \varphi$ then $t \models \psi$

It follows from the above definition that the empty state supports any formula φ. Thus, we may think of $\emptyset$ as the *inconsistent* state. The following two basic facts about support can be established by a straightforward induction on the complexity of φ:

Proposition 2.4 (Persistence)**.** If $s \models \varphi$ then for every $t \subseteq s$: $t \models \varphi$

Proposition 2.5 (Singleton states behave classically)**.** For any index w and formula φ:

$$\{w\} \models \varphi \iff w \models \varphi$$

where $w \models \varphi$ means: φ is classically true under the valuation w. In particular, $\{w\} \models \varphi$ or $\{w\} \models \neg\varphi$ for any formula φ.

It follows from definition 2.3 that the support-conditions for $\neg\varphi$ and $!\varphi$ are as follows.

Proposition 2.6 (Support for negation)**.**

1. $s \models \neg\varphi \quad \text{iff} \quad \forall w \in s : w \models \neg\varphi$
2. $s \models !\varphi \quad \text{iff} \quad \forall w \in s : w \models \varphi$

Proof. Clearly, since ! abbreviates double negation, item 2 is a particular case of item 1. To prove item 1, first suppose $s \models \neg\varphi$. Then for any $w \in s$ we have $\{w\} \models \neg\varphi$ by persistence, and thus $w \models \neg\varphi$ by proposition 2.5.
Conversely, if $s \not\models \neg\varphi$, then there must be $t \subseteq s$ with $t \models \varphi$ and $t \not\models \bot$. Since $t \not\models \bot$, $t \neq \emptyset$: thus, taken $w \in t$, by persistence and the classical behaviour of singleton states we have $w \models \varphi$. Since $w \in t \subseteq s$, it is not the case that $v \vdash \neg\varphi$ for all $v \in s$. □

The following construction will often be useful when dealing with cases where the set of propositional letters is infinite.

Definition 2.7. Let $\mathcal{P} \subseteq \mathcal{P}'$ be two sets of propositional letters. Then for any $\mathcal{P}'-$state s, the *restriction* of s to $\mathcal{P}$ is defined as $s{\restriction}_{\mathcal{P}} := \{w \cap \mathcal{P} \mid w \in s\}$.

The following fact, which can be established by a straightforward induction on the complexity of φ, says that whether or not a state s supports a formula φ only depends on the 'component' of s that is concerned with the letters in φ.

Proposition 2.8 (Restriction Invariance)**.** Let $\mathcal{P} \subseteq \mathcal{P}'$ be two sets of propositional letters. Then for any $\mathcal{P}'-$state s and any formula φ whose propositional letters are in $\mathcal{P}$:

$$s \models \varphi \quad \Longleftrightarrow \quad s{\restriction}_{\mathcal{P}} \models \varphi$$

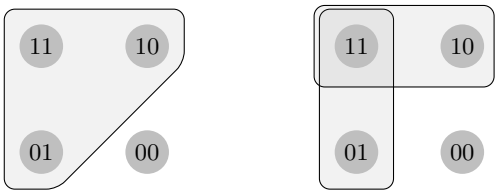
2.2 Possibilities, Propositions, and Truth-Sets

In terms of support, we define the *possibilities* for a sentence φ and the *proposition* expressed by φ. We think of the proposition expressed by φ as the *meaning* of φ in inquisitive semantics. We also define the *truth-set* of φ, which embodies the *classical meaning* of φ.

Definition 2.9 (Truth sets, possibilities, propositions)**.** Let φ be a formula.

1. A *possibility* for φ is a maximal state supporting φ, that is, a state that supports φ and is not properly included in any other state supporting φ.
2. The *proposition* expressed by φ, denoted by $[\varphi]$, is the set of possibilities for φ.
3. The *truth set* of φ, denoted by $|\varphi|$, is the set of indices where φ is classically true.

Notice that $|\varphi|$ is a state, while $[\varphi]$ is a set of states. The classical meaning of φ is the set of all indices that make φ true. In inquisitive semantics, meaning is defined in terms of support rather than in terms of truth. It may be expected, then, that the proposition expressed by φ would be defined as the set of all states supporting φ. Rather, though, it is defined as the set all *maximal* states supporting φ, that is, the set of all *possibilities* for φ. This is motivated by the fact that propositions are viewed as proposals, consisting of one or more alternative possibilities. If one state is included in another, we do not regard these two states as *alternatives*. This is why we are particularly interested in *maximal* states supporting a formula. Technically, however, the proposition expressed by φ still fully determines which states support φ and which states do not: the next result establishes that a state supports φ iff it is included in a possibility for φ.



(a) $|p \vee q|$ (b) $[p \vee q]$

Fig. 1 The truth-set of $p \vee q$, and the proposition it expresses.

Proposition 2.10 (Support and Possibilities)**.** For any state s and any formula φ:

$$s \models \varphi \quad \Longleftrightarrow \quad s \text{ is contained in a possibility for } \varphi$$

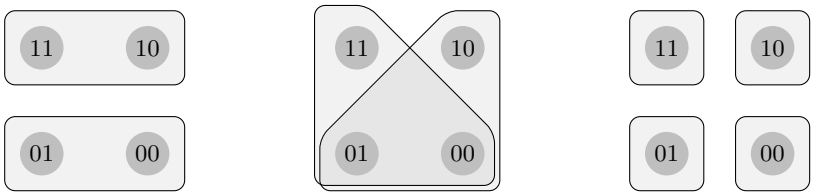
Proof. If $s \subseteq t$ and t is a possibility for φ, then by persistence $s \models \varphi$. For the converse, first consider the case in which the set $\mathcal{P}$ of propositional letters is finite. Then there are only finitely many states, and therefore if s supports φ, then obviously s must be contained in a *maximal* state supporting φ, i.e. in a possibility.
If $\mathcal{P}$ is infinite, given a $\mathcal{P}-$state $s \models \varphi$, consider its restriction $s{\restriction}_{\mathcal{P}_\varphi}$ to the (finite!) set $\mathcal{P}_\varphi$ of propositional letters occurring in φ. By proposition 2.8, $s{\restriction}_{\mathcal{P}_\varphi} \models \varphi$, and thus $s{\restriction}_{\mathcal{P}_\varphi} \subseteq t$ for some $\mathcal{P}_\varphi-$state t which is a possibility for φ.
Now, consider $t^+ := \{w \in \mathcal{I}_\mathcal{P} \mid w \cap \mathcal{P}_\varphi \in t\}$. For any $w \in s$ we have $w \cap \mathcal{P}_\varphi \in (s{\restriction}\mathcal{P}_\varphi) \subseteq t$, so $w \in t^+$ by definition of t^+; this proves that $s \subseteq t^+$. Moreover, we claim that t^+ is a possibility for φ.
First, since $t^+{\restriction}_{\mathcal{P}_\varphi} = t$ and $t \models \varphi$, it follows from proposition 2.8 that $t^+ \models \varphi$. Now, consider a state $u \supseteq t$ with $u \models \varphi$: then $u{\restriction}_{\mathcal{P}_\varphi} \supseteq t^+{\restriction}_{\mathcal{P}_\varphi} = t$ and moreover, again by proposition 2.8, $u{\restriction}_{\mathcal{P}_\varphi} \models \varphi$; but then, by the maximality of t it must be $u{\restriction}_{\mathcal{P}_\varphi} = t$. Now, for any $w \in u$, $w \cap \mathcal{P}_\varphi \in u{\restriction}_{\mathcal{P}_\varphi} = t$, so $w \in t^+$ by definition of t^+: hence, $u = t^+$. This proves that t^+ is indeed a possibility for φ. □

Example 2.11 (Disjunction)**.** Inquisitive semantics crucially differs from classical semantics in its treatment of disjunction. This is illustrated by figures 1(a) and 1(b). These figures assume that $\mathcal{P} = \{p, q\}$; index 11 makes both p and q true, index 10 makes p true and q false, etcetera. Figure 1(a) depicts the truth set—that is, the classical meaning—of $p \vee q$: the set of all indices that make either p or q, or both, true. Figure 1(b) depicts the proposition associated with $p \vee q$ in inquisitive semantics. It consists of two possibilities. One possibility is made up of all indices that make p true, and the other of all indices that make q true. So in the inquisitive setting, $p \vee q$ proposes two alternative ways of enhancing the common ground, and invites a response that is directed at choosing between these two alternatives.

As an immediate consequence of proposition 2.6, the possibilities for a (doubly) negated formula can be characterized as follows.

Proposition 2.12 (Negation)**.**

1. $[\neg\varphi] = \{|\neg\varphi|\}$
2. $[!\varphi] = \{|\varphi|\}$



(a) $?p$ (b) $p \rightarrow ?q$ (c) $?p \wedge ?q$

Fig. 2 A polar question, a conditional question, and a conjoined question.

2.3 Inquisitiveness and Informativeness

Recall that propositions are viewed as proposals to change the common ground of a conversation. If $[\varphi]$ contains more than one possibility, then we say that φ is *inquisitive*. If the proposal expressed by φ is not rejected, then the indices that are not included in any of the possibilities for φ will be eliminated. If there are such indices—that is, if the possibilities for φ do not cover the entire space—then we say that φ is *informative*.

Definition 2.13 (Inquisitiveness and informativeness)**.**

- φ is *inquisitive* iff $[\varphi]$ contains at least two possibilities;
- φ is *informative* iff $[\varphi]$ proposes to eliminate certain indices: $\bigcup[\varphi] \neq \mathcal{I}$

Definition 2.14 (Questions and assertions)**.**

- φ is a *question* iff it is not informative;
- φ is an *assertion* iff it is not inquisitive.

Definition 2.15 (Contradictions and tautologies)**.**

- φ is a *contradiction* iff it is only supported by the inconsistent state, i.e. iff $[\varphi] = \{\emptyset\}$
- φ is a *tautology* iff it is supported by all states, i.e. iff $[\varphi] = \{\mathcal{I}\}$

It is easy to see that a formula is a contradiction iff it is a classical contradiction. This does not hold for tautologies. Classically, a formula is tautological iff it is not informative. In the present framework, a formula is tautological iff it is neither informative nor inquisitive. Classical tautologies may well be inquisitive.

Example 2.16 (Questions)**.** Figure 2 depicts the propositions expressed by the polar question $?p$, the conditional question $p \rightarrow ?q$, and the conjoined question $?p \wedge ?q$. Recall that $?p$ abbreviates $p \vee \neg p$. So $?p$ is an example of a classical tautology that is inquisitive: it invites a choice between two alternatives, p and $\neg p$. As such, it reflects the essential function of polar questions in natural language.

Example 2.17 (Disjunction, continued)**.** It is clear from figure 1(b) that $p \vee q$ is *both* inquisitive *and* informative: $[p \vee q]$ consists of two possibilities, which, together, do not cover the set of all indices. This means that $p \vee q$ is *neither* a question *nor* an assertion.

The following proposition gives some sufficient syntactic conditions for being an assertion. The straightforward proof has been omitted.

Proposition 2.18. For any propositional letter p and formulas φ, ψ:

1. p is an assertion;

2. $\bot$ is an assertion;
3. if φ, ψ are assertions, then $\varphi \wedge \psi$ is an assertion;
4. if ψ is an assertion, then $\varphi \to \psi$ is an assertion.

Note that items 2 and 4 imply that any negation is an assertion, which we already knew from remark 2.12. Of course, a declarative $!\varphi$ is also always an assertion.

Using proposition 2.18 inductively we obtain the following corollary showing that disjunction is the only source of inquisitiveness in our propositional language.[1]

Corollary 2.19. Any disjunction-free formula is an assertion.

In inquisitive semantics, the informative content of a formula φ is captured by the union $\bigcup[\varphi]$ of all the possibilities for φ. For φ proposes to eliminate all indices that are not in $\bigcup[\varphi]$. In a classical setting, the informative content of φ is captured by $|\varphi|$. Hence, the following result can be read as stating that inquisitive semantics agrees with classical semantics as far as informative content is concerned.

Proposition 2.20. For any formula φ: $\bigcup[\varphi] = |\varphi|$.

Proof. According to proposition 2.5, if $w \in |\varphi|$, then $\{w\} \models \varphi$. But then, by proposition 2.10, $\{w\}$ must be included in some $t \in [\varphi]$, whence $w \in \bigcup[\varphi]$. Conversely, any $w \in \bigcup[\varphi]$ belongs to a possibility for φ, so by persistence and the classical behaviour of singletons we must have that $w \in |\varphi|$. □

We end this subsection with a definition of equivalence between two formulas, several characterizations of questions and assertions, and a remark about the behaviour of the operators ? and !.

Definition 2.21 (Equivalence)**.**
Two formulas φ and ψ are *equivalent*, $\varphi \equiv \psi$, iff $[\varphi] = [\psi]$.

It follows immediately from proposition 2.10 that $\varphi \equiv \psi$ just in case φ and ψ are supported by the same states.

Proposition 2.22 (Characterization of questions)**.**
For any formula φ, the following are equivalent:

1. φ is a question
2. φ is a classical tautology
3. $\neg\varphi$ is a contradiction
4. $\varphi \equiv ?\varphi$

Proof. Equivalence ($1 \Leftrightarrow 2$) follows from the definition of questions and proposition 2.20. ($2 \Leftrightarrow 3$) and ($4 \Rightarrow 3$) are immediate from the fact that a formula is a contradiction in the inquisitive setting just in case it is a classical contradiction. For ($3 \Rightarrow 4$), note that for any state s, $s \models ?\varphi$ iff $s \models \varphi$ or $s \models \neg\varphi$. This means that, if $\neg\varphi$ is a contradiction, $s \models ?\varphi$ iff $s \models \varphi$. In other words, $\varphi \equiv ?\varphi$. □

Note that an interrogative $?\varphi = \varphi \vee \neg\varphi$ is always a classical tautology, and therefore, by the equivalence ($1 \Leftrightarrow 2$), always a question. Furthermore, the equivalence ($1 \Leftrightarrow 4$) guarantees that $?\varphi \equiv ??\varphi$, which means that ? is idempotent.

[1] In the first-order case there will be a close similarity between disjunction and the existential quantifier, and the latter will be a source of inquisitiveness as well.

Proposition 2.23 (Characterization of assertions)**.**
For any formula φ, the following are equivalent:

1. φ is an assertion
2. if $s_j \models \varphi$ for all $j \in J$, then $\bigcup_{j\in J} s_j \models \varphi$
3. $|\varphi| \models \varphi$
4. $\varphi \equiv !\varphi$
5. $[\varphi] = \{|\varphi|\}$

Proof.

($1 \Rightarrow 2$) Suppose φ is an assertion and let t be the unique possibility for φ. If $s_j \models \varphi$ for all $j \in J$, then by proposition 2.10 each s_j must be a subsets of t, whence also $\bigcup_{j\in J} s_j \subseteq t$. Thus, by persistence, $\bigcup_{j\in J} s_j \models \varphi$.

($2 \Rightarrow 3$) By proposition 2.5, $\{w\} \models \varphi$ iff $w \in |\varphi|$. Then if φ satisfies condition (2), $|\varphi| = \bigcup_{w\in|\varphi|}\{w\} \models \varphi$.

($3 \Rightarrow 4$) Suppose $|\varphi| \models \varphi$; by proposition 2.10, $|\varphi|$ must be included in some possibility s for φ; but also, by corollary 2.20, $s \subseteq |\varphi|$, whence $|\varphi| = s \in [\varphi]$. Moreover, since any possibility for φ must be included in $|\varphi|$ we conclude that $|\varphi|$ must be the *unique* possibility for φ. Thus, $[\varphi] = \{|\varphi|\}$.

($4 \Leftrightarrow 5$) Since $[!\varphi] = \{|\varphi|\}$ (see remark 2.12), obviously $\varphi \equiv !\varphi \iff [\varphi] = \{|\varphi|\}$.

($5 \Rightarrow 1$) Immediate. □

Note that ($1 \Leftrightarrow 5$) states that a formula is an assertion if and only if its meaning consists of its classical meaning. In this sense, assertions behave classically. Also note that ($1 \Leftrightarrow 4$), together with the fact that $!\varphi$ is always an assertion, implies that $!\varphi \equiv !!\varphi$. That is, ! is idempotent.

The operators ! and ? work in a sense like projections on the 'planes' of assertions and questions, respectively. Moreover, the following proposition shows that the inquisitive meaning of a formula φ is completely determined by its 'purely informative component' $!\varphi$ and its 'purely inquisitive component' $?\varphi$.

Proposition 2.24 (Division in theme and rheme)**.** For any formula φ, $\varphi \equiv !\varphi \wedge ?\varphi$.

Proof. We must show that for any state s, $s \models \varphi$ iff $s \models !\varphi \wedge ?\varphi$. Suppose $s \models !\varphi \wedge ?\varphi$. Then, since $s \models ?\varphi$, s must support one of φ and $\neg\varphi$; but since $s \models \neg\neg\varphi$, s cannot support $\neg\varphi$. Thus, we have that $s \models \varphi$. The converse is immediate by the definitions of ! and ? and proposition 2.6. □

2.4 Support, Inquisitiveness, and Informativeness

The basic notion in the semantics, as we have set it up here, is the notion of support. In terms of support, we defined possibilities and propositions, and in terms of possibilities we defined the notions of inquisitiveness and informativeness. We have tried to make clear how possibilities, propositions, inquisitiveness, and informativeness should be thought of intuitively, but we have not said much as to how the notion of support itself should be interpreted. It is important to emphasize that support should *not* be thought of as specifying conditions under which an agent with information state s can *truthfully utter* a sentence φ (this is a common interpretation of the notion of support in

dynamic semantics, cf. Groenendijk *et al.*, 1996). Rather, in the present setting support should be thought of as specifying conditions under which a sentence φ is *insignificant* or *redundant* in a state s, in the sense that, given the information available in s, φ is neither informative nor inquisitive. This intuition can be made precise by defining notions of inquisitiveness and informativeness *relative to a state*.

Definition 2.25 (Relative semantic notions)**.** Let φ be a formula, and s a state. Then:

- A *possibility* for φ in s is a maximal substate of s supporting φ;
- φ is *inquisitive* in s iff there are at least two possibilities for φ in s;
- φ is *informative* in s iff there is at least one index in s that is not included in any possibility for φ in s.

These notions allow us to formally establish the connection between support on the one hand, and inquisitiveness and informativeness on the other. The straightforward proof is omitted.

Proposition 2.26 (Support, inquisitiveness, and informativeness)**.**
A state s supports a formula φ iff φ is neither inquisitive in s nor informative in s.

3 Inquisitive Logic

We are now ready to start investigating the logic that inquisitive semantics gives rise to. We begin by specifying the pertinent notions of entailment and validity.

Definition 3.1 (Entailment and validity)**.** A set of formulas Θ *entails* a formula φ in inquisitive semantics, $\Theta \models_{\mathsf{InqL}} \varphi$, if and only if any state that supports all formulas in Θ also supports φ. A formula φ is *valid* in inquisitive semantics, $\models_{\mathsf{InqL}} \varphi$, if and only if φ is supported by all states.

If no confusion arises, we will simply write $\models$ instead of $\models_{\mathsf{InqL}}$. We will also write $\psi_1, \ldots, \psi_n \models \varphi$ instead of $\{\psi_1, \ldots, \psi_n\} \models \varphi$. Note that, as expected, $\varphi \equiv \psi$ iff $\varphi \models \psi$ and $\psi \models \varphi$.

The intuitive interpretation of support in terms of insignificance or redundancy carries over to entailment: one can think of $\varphi \models \psi$ as saying that, whenever we are in a state where φ is redundant—i.e., neither informative nor inquisitive—ψ is as well. Or, in more dynamic terms, whenever we are in a state where the information provided by φ has been accommodated and the issue raised by φ has been resolved, ψ does not provide any new information and does not raise any new issue.

The following proposition states that if ψ is an assertion, inquisitive entailment boils down to classical entailment.

Proposition 3.2. If ψ is an assertion, $\varphi \models \psi$ iff $|\varphi| \subseteq |\psi|$.

Proof. Follows from proposition 2.23 and the definition of entailment. □

We have already seen that the ! operator turns any formula into an assertion. We are now ready to give a more precise characterization: for any formula φ, $!\varphi$ is *the most informative assertion entailed by* φ.

Proposition 3.3. For any formula φ and any assertion χ, $\varphi \models \chi \iff !\varphi \models \chi$.

Proof. Fix a formula φ and an assertion χ. The right-to-left implication is obvious, since it is clear from proposition 2.6 that $\varphi \models !\varphi$. For the converse direction, suppose $\varphi \models \chi$. Any possibility $s \in [\varphi]$ supports φ and therefore also χ, whence by proposition 2.10 it must be included in a possibility for χ, which must be $|\chi|$ by proposition 2.23 on assertions. But then also $|\varphi| = \bigcup[\varphi] \subseteq |\chi|$ whence $!\varphi \models \chi$ by proposition 3.2. □

Most naturally, since a question does not provide any information, it cannot entail informative formulas.

Proposition 3.4. If φ is a question and $\varphi \models \psi$, then ψ must be a question as well.

Proof. If φ is a question, it must be supported by each singleton state. If moreover $\varphi \models \psi$, then ψ must also be supported by each singleton state. But then, since singletons behave like indices, ψ must be a classical tautology, that is, a question. □

Definition 3.5 (Logic)**.** Inquisitive logic, InqL, is the set of formulas that are valid in inquisitive semantics.

Clearly, a formula is valid in inquisitive semantics if and only if it is both a classical tautology *and* an assertion. Thus, InqL coincides with classical logic as far as assertions are concerned: in particular, it agrees with classical logic on the whole disjunction-free fragment of the language.

Remark 3.6. Although InqL is closed under the *modus ponens* rule, it is *not* closed under uniform substitution. For instance, $\neg\neg p \to p \in \mathsf{InqL}$ for all proposition letters, but $\neg\neg(p \vee q) \to (p \vee q) \notin \mathsf{InqL}$. We will return to this feature of the logic below, especially in section 4 and section 9.

3.1 Disjunction Property, Deduction Theorem, Compactness, and Decidability

We proceed by establishing a few basic properties of inquisitive logic and entailment. The first observation is an immediate consequence of the fact that support is persistent.

Proposition 3.7. For any formula φ, $\varphi \in \mathsf{InqL} \iff \mathcal{I} \models \varphi$

Corollary 3.8 (Disjunction Property)**.** InqL has the *disjunction property*. That is, whenever a disjunction $\varphi \vee \psi$ is in InqL, at least one of φ and ψ is in InqL as well.

Proposition 3.9 (Deduction theorem)**.** For any formulae $\theta_1, \ldots, \theta_n, \varphi$:

$$\theta_1, \ldots, \theta_n \models \varphi \quad \iff \quad \models \theta_1 \wedge \cdots \wedge \theta_n \to \varphi$$

Proof. $\theta_1, \ldots, \theta_n \models \varphi$

$\iff$ for any $s \in \mathcal{S}$, if $s \models \theta_i$ for $1 \leq i \leq n$, then $s \models \varphi$

$\iff$ for any $s \in \mathcal{S}$, if $s \models \theta_1 \wedge \cdots \wedge \theta_n$, then $s \models \varphi$

$\iff \mathcal{I} \models \theta_1 \wedge \cdots \wedge \theta_n \to \varphi$

$\iff \theta_1 \wedge \cdots \wedge \theta_n \to \varphi \in \mathsf{InqL}$ □

Theorem 3.10 (Compactness)**.** For any set Θ and any formula φ, if $\Theta \models \varphi$ then there is a *finite* set $\Theta_0 \subseteq \Theta$ such that $\Theta_0 \models \varphi$.

Proof. Since our set $\mathcal{P}$ of propositional letters is countable, so must be Θ, so we can write $\Theta = \{\theta_k \mid k \in \omega\}$. Now for any $k \in \omega$, let $\gamma_k = \theta_0 \wedge \cdots \wedge \theta_k$, and define $\Gamma = \{\gamma_k \mid k \in \omega\}$. It is clear that Γ and Θ are equivalent, in the sense that for any state s, $s \models \Gamma \iff s \models \Theta$, so we have $\Gamma \models \varphi$. Moreover, for $k \geq k'$ we have $\gamma_k \models \gamma_{k'}$. If we can show that there is a formula $\gamma_k \in \Gamma$ such that $\gamma_k \models \varphi$, then this will mean that $\{\theta_0, \ldots, \theta_k\} \models \varphi$, and since $\{\theta_0, \ldots, \theta_k\}$ is a finite subset of Θ we will be done.
For any $k \in \omega$ let $\mathcal{P}_k$ be the set of propositional letters occurring in φ or in γ_k. By the definition of the formulas γ_k, it is clear that for $k \leq k'$ we have $\mathcal{P}_k \subseteq \mathcal{P}_{k'}$.
Now, towards a contradiction, suppose there is no $k \in \omega$ such that $\gamma_k \models \varphi$. Define $L_k := \{t \mid t \text{ is a } \mathcal{P}_k\text{−state with } t \models \gamma_k \text{ but } t \not\models \varphi\}$: our assumption amounts to saying that $L_k \neq \emptyset$ for all k. Then put $L := \{\emptyset\} \cup \biguplus_{k\in\omega} L_k$. Define a relation R on L by putting:

- $\emptyset R t$ iff $t \in L_0$;
- sRt iff $s \in L_k$, $t \in L_{k+1}$ and $t{\restriction}_{\mathcal{P}_k} = s$.

Now, consider $t \in L_{k+1}$. This means that $t \models \gamma_{k+1}$ and $t \not\models \varphi$; as $\gamma_{k+1} \models \gamma_k$, we also have $t \models \gamma_k$. But then, since both γ_k and φ only use propositional letters from $\mathcal{P}_k$, by proposition 2.8 we have $t{\restriction}_{\mathcal{P}_k} \models \gamma_k$ and $t{\restriction}_{\mathcal{P}_k} \not\models \varphi$, which means that $t{\restriction}_{\mathcal{P}_k} \in L_k$.
From this it follows that (L, R) is a connected graph and thus clearly a tree with root $\emptyset$. Since L is a disjoint union of infinitely many non-empty sets, it must be infinite. On the other hand, by definition of R, all the successors of a state $s \in L_k$ are $\mathcal{P}_{k+1}$-states, and there are only finitely many of those as $\mathcal{P}_{k+1}$ is finite. Therefore, the tree (L, R) is finitely branching.
By König's lemma, a tree that is infinite and finitely branching must have an infinite branch. This means that there must be a sequence $\langle t_k \mid k \in \omega\rangle$ of states in L such that for any k, $t_{k+1}{\restriction}_{\mathcal{P}_k} = t_k$. This naturally defines a $\mathcal{P}$-state that is the "limit" of the sequence. Precisely, this state is:

$$t = \{w \in \wp(\mathcal{P}) \mid \text{there are } w_k \in t_k \text{ with } w_{k+1}{\restriction}_{\mathcal{P}_k} = w_k \text{ and } w = \bigcup_{k\in\omega} w_k\}$$

It is easy to check that for any k, $t{\restriction}_{\mathcal{P}_k} = t_k$. Now, for any natural k, since $t{\restriction}_{\mathcal{P}_k} = t_k \models \gamma_k$, by proposition 2.8 we have $t \models \gamma_k$; hence, $t \models \Gamma$. On the other hand, for the same reason, since $t{\restriction}_{\mathcal{P}_0} = t_0 \not\models \varphi$, also $t \not\models \varphi$.
But this contradicts the fact that $\Gamma \models \varphi$. So for some k we must have $\gamma_k \models \varphi$. □

Remark 3.11 (Decidability)**.** InqL is clearly decidable: to determine whether a formula φ is in InqL, by propositions 2.8 and 3.7 we only have to test whether $\mathcal{I}_{\{p_1,\ldots,p_n\}}$ supports φ, where $p_1, \ldots, p_n$ are the propositional letters in φ. This is a finite procedure since $\mathcal{I}_{\{p_1,\ldots,p_n\}}$ is finite and has only finitely many substates which have to be checked to determine support for implications.

3.2 Disjunctive Negative Translation and Expressive Completeness

In this section, we observe that a formula can always be rewritten as a disjunction of negations, preserving logical equivalence with respect to inquisitive semantics. This observation will lead to a number of expressive completeness results. It will also play a crucial role later on in establishing completeness results, and in showing that InqL is isomorphic to the disjunctive-negative fragment of IPL.

We start by defining the *disjunctive negative translation* $\text{DNT}(\varphi)$ of a formula φ.

Definition 3.12 (Disjunctive Negative Translation)**.**

1. $\textsc{dnt}(p) = \neg\neg p$
2. $\textsc{dnt}(\bot) = \neg\neg\bot$
3. $\textsc{dnt}(\psi \vee \chi) = \textsc{dnt}(\psi) \vee \textsc{dnt}(\chi)$
4. $\textsc{dnt}(\psi \wedge \chi) = \bigvee\{\neg(\psi_i \vee \chi_j) \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$

   where:
   - $\textsc{dnt}(\psi) = \neg\psi_1 \vee \ldots \vee \neg\psi_n$
   - $\textsc{dnt}(\chi) = \neg\chi_1 \vee \ldots \vee \neg\chi_m$
5. $\textsc{dnt}(\psi \to \chi) = \bigvee_{k_1,\ldots,k_n}\{\neg\neg \bigwedge_{1\leq i\leq n}(\chi_{k_i} \to \psi_i) \mid 1 \leq k_j \leq m\}$

   where:
   - $\textsc{dnt}(\psi) = \neg\psi_1 \vee \ldots \vee \neg\psi_n$
   - $\textsc{dnt}(\chi) = \neg\chi_1 \vee \ldots \vee \neg\chi_m$

Proposition 3.13. For any φ, $\varphi \equiv_{\mathsf{InqL}} \textsc{dnt}(\varphi)$.

Proof. By induction on φ. □

We skip over the details of the proof here. However, in section 3.6 we will see that, given some auxiliary results, a close examination of what is needed exactly to prove proposition 3.13 instantly yields a range of interesting completeness results.

Note that the map $\textsc{dnt}$ always returns a disjunction of negations, so we immediately have the following corollary.

Corollary 3.14. Any formula is equivalent to a disjunction of negations.

In particular, any formula is equivalent to a disjunction of assertions. This perfectly matches our intuitive understanding that meanings in inquisitive semantics are sets of *alternatives*, which are incomparable classical meanings. Classical meanings are expressed by assertions (and thus always expressible by negations) while disjunction is the source of alternativehood, in the sense that a disjunction applied to two incomparable classical meanings yields the proposition consisting of those two classical meanings as alternatives.

Additionally, note that since any negation behaves classically in inquisitive semantics, in the scope of a negation we can always safely substitute classically equivalent subformulas. Since the set of connectives $\{\neg, \vee\}$ is complete in classical logic, given a formula $\varphi \equiv \neg\chi_1 \vee \cdots \vee \neg\chi_n$ we can always substitute each χ_k by a classically equivalent formula ${\chi_k}'$ using only disjunction and negation without altering the meaning of the formula, thus getting $\varphi \equiv \neg{\chi_1}' \vee \cdots \vee \neg{\chi_n}'$. This proves the following corollary.

Corollary 3.15 (Expressive completeness of $\{\neg, \vee\}$)**.** Any formula is equivalent to a formula containing only disjunctions and negations.

Now consider an assertion χ. Since the set of connectives $\{\neg, \wedge\}$ is complete in classical logic, let χ' be a formula classically equivalent to χ which only contains negations and conjunctions: the classical equivalence of χ and χ' amounts to $|\chi| = |\chi'|$. Now, χ' is an assertion by corollary 2.19, whence using proposition 2.23 we have $[\chi] = \{|\chi|\} = \{|\chi'|\} = [\chi']$, i.e., $\chi \equiv \chi'$. Thus, we have the following corollary, stating that the assertive fragment coincides—up to equivalence—with the $\{\neg, \wedge\}$-fragment of the language.

Corollary 3.16 (Expressive completeness of $\{\neg, \wedge\}$ for assertions.)**.** A formula is an assertion iff it is equivalent to a formula containing only conjunctions and negations.

3.3 Inquisitive semantics and intuitionistic Kripke semantics

We now turn to the connection between inquisitive logic and intuitionistic logic.[2] This connection is suggested by the existence of a striking analogy between inquisitive and intuitionistic semantics. Both can be conceived of in terms of information and information growth. In inquisitive semantics, a formula is evaluated with respect to a state. Such a state can be thought of as an information state. Whether a certain state s supports a formula φ may depend not only on the information available in s, but also on the information that may *become* available. Formally, support is partly defined in terms of *subsets* of s. These subsets can be seen as possible future information states.

Similarly, in intuitionistic semantics, a formula is evaluated with respect to a point in a Kripke model, which can also be thought of as an information state. Each point comes equipped with a set of future points, called successors. Whether a point u in a model M satisfies a formula φ may depend not only on the information available at u, but also on the information that may become available. Formally, satisfaction at u is partly defined in terms of points in M that are accessible from u.

This informal analogy can be made precise: in fact, inquisitive semantics amounts to intuitionistic semantics on a suitable Kripke model.

Definition 3.17 (Kripke model for inquisitive semantics)**.** The Kripke model for inquisitive semantics is the model $M_I = \langle W_I, \supseteq, V_I \rangle$ where $W_I := \mathcal{S} - \{\emptyset\}$ is the set of all non-empty states and the valuation V_I is defined as follows: for any letter p, $V_I(p) = \{s \in W_I \mid s \models p\}$.

Observe that M_I is a Kripke model for intuitionistic logic. For, the relation $\supseteq$ is clearly a partial order. Moreover, suppose $s \supseteq t$ and $s \in V_I(p)$: this means that $s \models p$, and so by persistence $t \models p$, which amounts to $t \in V_I(p)$. So the valuation V_I is persistent. The next lemma shows that the two semantics coincide on every non-empty state.

Proposition 3.18 (Inquisitive support coincides with Kripke satisfaction on M_I)**.** For every formula φ and every non-empty state s:

$$s \models \varphi \iff M_I, s \Vdash \varphi$$

Proof. Straightforward, by induction on φ. The inductive step for implication uses the fact that an implication cannot be falsified by the empty state, as the latter supports all formulas, so that restricting the semantics to non-empty states does not make a difference. □

This simple observation already shows that the logic InqL contains intuitionistic propositional logic IPL. For suppose that $\varphi \notin \mathsf{InqL}$. Then there must be a non-empty state s such that $s \not\models \varphi$. But then we also have that $M_I, s \nVdash \varphi$, which means that $\varphi \notin \mathsf{IPL}$.

On the other hand, InqL is contained in classical propositional logic CPL, because any formula that is not a classical tautology is falsified by a singleton state in inquisitive semantics. So we have:

$$\mathsf{IPL} \subseteq \mathsf{InqL} \subseteq \mathsf{CPL}$$

Moreover, both inclusions are strict: for instance, $p \vee \neg p$ is in CPL but not in InqL, while $\neg\neg p \to p$ is in InqL but not in IPL.

[2] Our investigation of this connection was inspired by Groenendijk (2008a) and van Benthem (2009).

Our next task is to investigate exactly where InqL sits between IPL and CPL. Ultimately, our result will be that there is a whole range of intermediate logics whose 'negative variant' coincides with InqL. In order to get to this result, let us first recall some basic features of intermediate logics, and define precisely what we mean by the negative variant of a logic.

3.4 Intermediate Logics and Negative Variants

Recall that an *intermediate logic* is defined as a consistent set of formulae that contains IPL and is closed under the rules of modus ponens and uniform substitution, where *consistent* simply means 'not containing $\bot$'.

Intermediate logics ordered by inclusion form a complete lattice whose meet operation amounts to intersection and whose join operation, also called *sum*, is defined as follows: if Λ_i, $i \in I$ is a family of intermediate logics, then $\Sigma_{i\in I}\Lambda_i$ is the logic axiomatized by $\bigcup_{i\in I}\Lambda_i$, that is, the closure of $\bigcup_{i\in I}\Lambda_i$ under modus ponens and uniform substitution. The sum of *two* intermediate logic Λ and Λ' is denoted by $\Lambda + \Lambda'$.

In our investigation, we will meet several logics, beginning with InqL itself, that are *not* closed under uniform substitution. We shall refer to such logics as *weak intermediate logics*.

Definition 3.19. A weak intermediate logic is a set L of formulae closed under modus ponens and such that $\mathsf{IPL} \subseteq L \subseteq \mathsf{CPL}$.

Weak intermediate logics ordered by inclusion form a complete lattice as well, where again meet is intersection and the join (or sum) of a family is the weak logic axiomatized by the union, i.e. the closure of the union under modus ponens.

If L is a weak intermediate logic, we write $\varphi \equiv_L \psi$ just in case $\varphi \leftrightarrow \psi \in L$.

Definition 3.20. Let K be a class of Kripke models (resp., frames). If Θ is a set of formulae and φ is a formula, we write $\Theta \models_K \varphi$ just in case any point in any model in K (resp., any point in any model based on any frame in K), that satisfies all formulas in Θ, also satisfies φ. We denote by $\mathsf{Log}(K)$ the set of formulae that are valid on each model (frame) in K, that is: $\mathsf{Log}(K) = \{\varphi \mid \ \models_K \varphi\}$.

It is straightforward to check that for any class K of Kripke frames, $\mathsf{Log}(K)$ is an intermediate logic.

Notation. For any formula φ, we denote by φ^n the formula obtained from φ by replacing any occurrence of a propositional letter with its negation.

Definition 3.21 (Negative variant of a logic)**.** If Λ is an intermediate logic, we define its negative variant Λ^n as:

$$\Lambda^n = \{\varphi \mid \varphi^n \in \Lambda\}$$

In general, Λ^n will not be an intermediate logic: in fact, for any intermediate logic Λ we have $\neg\neg p \to p \in \Lambda^n$ for any proposition letter p. So Λ^n will not be closed under uniform substitution unless $\Lambda^n = \mathsf{CPL}$, where CPL denotes classical logic.

It is, however, the case that Λ^n is always a weak intermediate logic containing Λ; this is the content of the following remark.

Remark 3.22. For any intermediate logic Λ, its negative variant Λ^n is a weak intermediate logic including Λ.

Proof. Fix an intermediate logic Λ. Since Λ is closed under uniform substitution, $\varphi \in \Lambda$ implies $\varphi^n \in \Lambda$ and so $\varphi \in \Lambda^n$. This shows $\Lambda \subseteq \Lambda^n$.

Moreover, if both φ and $\varphi \to \psi$ belong to Λ^n, then both φ^n and $(\varphi \to \psi)^n = \varphi^n \to \psi^n$ are in Λ which is closed under modus ponens; therefore, $\psi^n \in \Lambda$, which means that $\psi \in \Lambda^n$. This shows that Λ^n is closed under modus ponens.

Finally, if $\varphi \in \Lambda^n$ then $\varphi^n \in \Lambda \subseteq \mathsf{CPL}$; but then, since CPL is substitution-closed, $\varphi^{nn} \in \mathsf{CPL}$ and therefore also $\varphi \in \mathsf{CPL}$ because the double negation law holds in CPL. This shows that $\Lambda^n \subseteq \mathsf{CPL}$ and therefore that Λ^n is indeed a weak intermediate logic. □

The following observation will turn out useful below.

Remark 3.23. If a logic Λ has the disjunction property, then so does Λ^n.

Proof. If $\varphi \vee \psi \in \Lambda^n$, then $\varphi^n \vee \psi^n \in \Lambda$; thus, by the disjunction property, at least one of φ^n and ψ^n is in Λ, which means that at least one of φ and ψ is in Λ^n. □

Definition 3.24 (Negative valuations)**.** Let F be an intuitionistic frame. A valuation V is called *negative* in case for any point w in F and for any proposition letter p:

$$(F, V), w \Vdash p \iff (F, V), w \Vdash \neg\neg p$$

We will call a model *negative* in case its valuation is negative. Observe that if M is a negative model, for any point w and formula φ we have $M, w \Vdash \varphi \iff M, w \Vdash \varphi^{nn}$.

Definition 3.25 (Negative variant of a model)**.** If $M = (W, R, V)$ is a Kripke model, we define the negative variant M^n of M to be model $M^n = (M, R, V^n)$ where

$$V^n(p) := \{w \in W \mid M, w \Vdash \neg p\}$$

that is, V^n makes a propositional letter true precisely where its negation was true in the original model.

A straightforward inductive proof yields the following result.

Proposition 3.26. For any model M, any point w and formula φ:

$$M, w \Vdash \varphi^n \iff M^n, w \Vdash \varphi$$

Remark 3.27. For any model M, its negative variant M^n is a negative model.

Proof. Take any point w of M and formula φ. According to the previous proposition and recalling that in intuitionistic logic triple negation is equivalent to single negation, we have $M^n, w \Vdash p \iff M, w \Vdash \neg p \iff M, w \Vdash \neg\neg\neg p \iff M^n, w \Vdash \neg\neg p$. □

Definition 3.28. Let K be a class of intuitionistic Kripke frames. Then we denote by $\mathbf{n}K$ the class of negative K-models, i.e., negative Kripke models whose frame is in K.

Proposition 3.29. For any class K of Kripke frames, $\mathsf{Log}(\mathbf{n}K) = \mathsf{Log}(K)^n$.

Proof. If $\varphi \notin \mathsf{Log}(K)^n$, i.e. if $\varphi^n \notin \mathsf{Log}(K)$, then there must be a K-model M (i.e., a model based on a K-frame) and a point w such that $M, w \nVdash \varphi^n$. But then, by proposition 3.26 we have $M^n, w \nVdash \varphi$, and thus $\varphi \notin \mathsf{Log}(\mathbf{n}K)$ since M^n is a negative K-model.

Conversely, if $\varphi \notin \mathsf{Log}(\mathbf{n}K)$, let M be a negative K-model and w a point in M with $M, w \nVdash \varphi$. Then since M is negative, $M, w \nVdash \varphi^{nn}$. Therefore, by proposition 3.26, $M^n, w \nVdash \varphi^n$. But M^n shares the same frame of M, which is a K-frame: so $\varphi^n \notin \mathsf{Log}(K)$, that is, $\varphi \notin \mathsf{Log}(K)^n$. □

The following result states that for any intermediate logic Λ, Λ^n is axiomatized by a system having Λ and all the atomic double negation formulas $\neg\neg p \to p$ as axioms, and modus ponens as unique inference rule.

Proposition 3.30. If Λ is an intermediate logic, Λ^n is the smallest weak intermediate logic containing Λ and the atomic double negation axiom $\neg\neg p \to p$ for each propositional letter p.

Proof. We have already observed (see remark 3.4) that Λ^n is a weak intermediate logic containing Λ; moreover, for any letter p we have $\neg\neg\neg p \to \neg p \in \mathsf{IPL} \subseteq \Lambda$, so each atomic double negation formula is in Λ^n.

To see that Λ^n is *the smallest* such logic, let Λ' be another weak logic containing Λ and the atomic double negation axioms. Consider $\varphi \in \Lambda^n$: this means that $\varphi^n \in \Lambda$. But clearly, φ is derivable by modus ponens from φ^n and the atomic double negation axioms for letters in φ: hence, as Λ' contains Λ and the atomic double negation formulas and it is closed under modus ponens, $\varphi \in \Lambda'$. Thus, $\Lambda^n \subseteq \Lambda'$. □

With slight abuse of notation, we will henceforth identify Λ^n not only with a set of formulas, but also with the following derivation system:

Axioms:
- all formulas in Λ
- $\neg\neg p \to p$ for all proposition letters $p \in \mathcal{P}$

Rules:
- modus ponens

If Θ is a set of formulae and φ is a formula, we will write $\Theta \vdash_{\Lambda^n} \varphi$ in case φ is derivable from the set of assumptions Θ in the system Λ^n.

3.5 Disjunction Property + Disjunctive Negative Translation = InqL

We are now ready to connect some of the notions introduced in previous subsections. The following theorem characterizes InqL as the unique weak intermediate logic that has the disjunction property and preserves logical equivalence under disjunctive negation translation (as defined in section 3.2).

Theorem 3.31. Let L be a weak intermediate logic. If $\varphi \equiv_L \textsc{dnt}(\varphi)$ for all formulas φ, then $\mathsf{InqL} \subseteq L$. If, additionally, L has the disjunction property, then $L = \mathsf{InqL}$.

Proof. Let L be a weak intermediate logic for which any formula φ is equivalent to $\textsc{dnt}(\varphi)$. Suppose $\varphi \in \mathsf{InqL}$. Then $\textsc{dnt}(\varphi) \in \mathsf{InqL}$. Write $\textsc{dnt}(\varphi) = \neg\nu_1 \vee \cdots \vee \neg\nu_k$: since InqL has the disjunction property, we must have $\neg\nu_i \in \mathsf{InqL} \subseteq \mathsf{CPL}$ for some $1 \leq i \leq k$. But IPL coincides with CPL as far as negations are concerned (cf. Chagrov and Zakharyaschev, 1997), so $\neg\nu_i \in \mathsf{IPL} \subseteq L$. Hence, obviously, $\textsc{dnt}(\varphi) \in L$, and since $\varphi \equiv_L \textsc{dnt}(\varphi)$, also $\varphi \in L$. This shows that $\mathsf{InqL} \subseteq L$.

Moreover, suppose L also has the disjunction property. Consider a formula $\varphi \in L$: since $\varphi \equiv_L \textsc{dnt}(\varphi)$ we have $\textsc{dnt}(\varphi) \in L$. But L has the disjunction property and therefore again, using the same notation as above, $\neg\nu_i \in \Lambda^n$ for some i. Then again because all weak intermediate logics agree on negations, $\neg\nu_i \in \mathsf{InqL}$, whence $\textsc{dnt}(\varphi) \in \mathsf{InqL}$ and also $\varphi \in \mathsf{InqL}$. This proves that $L \subseteq \mathsf{InqL}$. □

3.6 Axiomatizing Inquisitive Logic

Given that InqL is the only weak intermediate logic with the disjunction property that preserves logical equivalence under $\textsc{dnt}$, it is natural to ask next what exactly is required, in terms of axioms, in order to preserve logical equivalence under $\textsc{dnt}$. Answering this question will directly lead to an axiomatization of InqL.

In order to identify the relevant axioms, let us go back to the proof of proposition 3.13, which stated that $\varphi \equiv_{\mathsf{InqL}} \textsc{dnt}(\varphi)$ for any φ. The proof is by induction on φ. The atomic case amounts to the validity of the atomic double negation axioms. The inductive step for disjunction is trivial, while the one for conjunction follows from the fact that $\mathsf{IPL} \subseteq \mathsf{InqL}$, which means that intuitionistic equivalences (like instances of the distributive laws) hold in the inquisitive setting.

Finally, for the inductive step for implication we need—in addition to some intuitionistically valid equivalences—the following equivalence:

$$\Big(\neg\chi \to \bigvee_{1\leq i\leq k} \neg\psi_i\Big) \quad \equiv_{\mathsf{InqL}} \quad \bigvee_{1\leq i\leq k} (\neg\chi \to \neg\psi_i)$$

for all formulas $\chi, \psi_1, \ldots, \psi_k$. Since the right-to-left entailment already holds intuitionistically, what is needed more specifically is that any substitution instance of each of the following formulas be valid in InqL:

$$\mathtt{ND}_k \qquad \big(\neg p \to \bigvee_{1\leq i\leq k} \neg q_i\big) \quad \longrightarrow \quad \bigvee_{1\leq i\leq k} (\neg p \to \neg q_i)$$

Thus—besides intuitionistic validities—we need all instances of $\mathtt{ND}_k$, $k \in \omega$, and all the atomic double negation axioms $\neg\neg p \to p$ in order to preserve logical equivalence under $\textsc{dnt}$. Any system containing those axioms and equipped with the modus ponens rule will be able to prove the equivalence between a formula φ and its translation $\textsc{dnt}(\varphi)$. This suffices to prove proposition 3.32, where ND is the intermediate logic axiomatized by the axioms $\mathtt{ND}_k$, $k \in \omega$.

Proposition 3.32. For any logic $\Lambda \supseteq \mathsf{ND}$ and any formula φ, $\varphi \equiv_{\Lambda^n} \textsc{dnt}(\varphi)$.

This proposition immediately yields a whole range of intermediate logics whose negative variant coincides with inquisitive logic.

Theorem 3.33 (Completeness theorem)**.** $\Lambda^n = \mathsf{InqL}$ for any logic $\Lambda \supseteq \mathsf{ND}$ with the disjunction property.

Proof. Let Λ be an extension of ND with the disjunction property. Then according to proposition 3.32 we have $\varphi \equiv_{\Lambda^n} \mathrm{DNT}(\varphi)$ for all φ; moreover, Λ^n has the disjunction property (see remark 3.23). Hence by theorem 3.31 we have $\Lambda^n = \mathsf{InqL}$. □

The logic ND has been studied by (Maksimova, 1986). Among other things, she shows (1) that ND has the disjunction property, and (2) that the maximal intermediate logic with the disjunction property containing ND is ML, an intermediate logic introduced by Medvedev (1962), also known as 'the logic of finite problems'. Maximova also remarks that KP, a well-known intermediate logic introduced by Kreisel and Putnam (1957), is one of the logics in between ND and ML which has the disjunction property. This immediately gives us three concrete axiomatizations of InqL.

Corollary 3.34. $\mathsf{ND}^n = \mathsf{KP}^n = \mathsf{ML}^n = \mathsf{InqL}$.

Medvedev's logic will be discussed in more detail in the next section, which investigates the schematic fragment of InqL. The completeness theorem established above will be further strengthened in section 5. There we will see that the negative variant of an intermediate logic Λ coincides with InqL *if and only if* $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$.

4 The Schematic Fragment of Inquisitive Logic

We have already remarked that inquisitive logic is not closed under uniform substitution; it is natural to ask, then, what the schematic fragment of InqL is. In this section we will address this issue and we will find that this fragment in fact coincides with Medvedev's logic of finite problems.

Definition 4.1 (Schematic fragment of InqL)**.** We denote by $\mathrm{Sch}(\mathsf{InqL})$ the set of formulae that are schematically valid in InqL, i.e., those formulae φ such that $\varphi^* \in \mathsf{InqL}$ for any substitution instance φ^* of φ.

Notice that $\mathrm{Sch}(\mathsf{InqL})$ is the greatest intermediate logic included in InqL.

Definition 4.2 (Medvedev frames)**.** A Medvedev frame consists of all the non-empty subsets of some finite set X, ordered by the superset relation. In other words, a Medvedev frame is a frame of the shape $(\wp(X) - \{\emptyset\}, \supseteq)$, where X is some finite set. The class of Medvedev frames will be denoted by **Med**.

Notice that the frame F_I underlying the Kripke model for inquisitive semantics is $(\wp\wp(\mathcal{P}) - \{\emptyset\}, \supseteq)$. So F_I is a Medvedev frame whenever the set of proposition letters $\mathcal{P}$ is finite.

Definition 4.3 (Medvedev Logic)**.** Medvedev logic is the logic of the class **Med** of Medvedev frames: $\mathsf{ML} := \mathsf{Log}(\mathbf{Med})$. Notice that ML is an intermediate logic.

The following theorem establishes the main result of this section, namely that the schematic fragment of InqL coincides with Medvedev logic. There is, however, one subtlety that should be remarked: whereas we assumed so far that the set of atomic proposition letters $\mathcal{P}$ may be finite or countably infinite, it is at this stage important to assume that $\mathcal{P}$ is in fact countably infinite.

Theorem 4.4. $\mathrm{Sch}(\mathsf{InqL})=\mathsf{ML}$.

In order to proof this theorem, we need the following lemma.

Lemma 4.5. For any negative Medvedev model M, there is a p-morphism η from M to the Kripke model for inquisitive logic M_I.

Proof. Let $M = (W, R, V)$ be a negative Medvedev model. For any endpoint e of M, denote by i_e the valuation $i_e = \{p \in P \mid e \in V(p)\}$ consisting of those letters true at e. For any point w in M, let E_w denote the set of endpoints accessible from w in M. Define the candidate p-morphism η as follows:

For any $w \in W$, $\eta(w) = \{i_e \mid e \in E_w\}$.

First, note that any point in M can see at least one endpoint. This means that for any $w \in W$, we have $E_w \neq \emptyset$, and therefore $\eta(w) \neq \emptyset$. This insures that $\eta(w) \in W_I$, so that the map $\eta : W \to W_I$ is well-defined. It remains to check that η is a p-morphism. Fix any $w \in W$:

- **Proposition Letters**. Take any proposition letter p. If $M, w \Vdash p$, then by persistence we have $M, e \Vdash p$ for any $e \in E_w$; this implies that $p \in i$ for any index $i \in \eta(w)$ and so $\eta(w) \models p$, whence $M_I, \eta(w) \Vdash p$.
  Conversely, suppose $M, w \nVdash p$. Then since the valuation V is negative, $M, w \nVdash \neg\neg p$, so there must be a successor v of w with $M, v \Vdash \neg p$. M is finite, so E_v is non-empty. Take a point $e \in E_v$. Then, by persistence, $M, e \Vdash \neg p$, whence $p \notin i_e$. But, by transitivity of R, we have that $e \in E_w$, so $i_e \in \eta(w)$. Thus $\eta(w) \not\models p$, whence $M_I, \eta(w) \nVdash p$.
- **Forth Condition**. Suppose wRv: then since our accessibility relation is transitive, $E_w \supseteq E_v$ and thus also $\eta(w) \supseteq \eta(v)$.
- **Back Condition**. Suppose $\eta(w) \supseteq t$: we must show that there is some successor v of w such that $\eta(v) = t$.
  Since t is a non-empty subset of $\eta(w) = \{i_e \mid e \in E_w\}$, there must be some non-empty subset $E_* \subseteq E_w$ such that $t = \{i_e \mid e \in E_*\}$. We must show, then, that there is a successor v of w in M whose terminal successors are exactly those in E_*. Recall that M is based on a frame that consists of all the non-empty subsets of some finite set X, ordered by the superset relation. In particular, all the endpoints in M are singleton subsets of X, and for any set of endpoints E, there is a point, namely $\bigcup E$, whose terminal successors are exactly the ones in E.
  Thus, for v we can take $\bigcup E_*$. Then, $E_v = E_*$, and $\eta(v) = t$. □

Proof of theorem 4.4. Suppose $\varphi \notin \mathrm{Sch}(\mathsf{InqL})$: then there is a substitution instance φ^* of φ such that $\varphi^* \notin \mathsf{InqL}$. But then it follows from proposition 2.8 that φ^* can be falsified in a point of the model M_I for inquisitive semantics relative to the finite set of propositional letters P_{φ^*}; and since this model is a Medvedev model, $\varphi^* \notin \mathsf{ML}$. But then, as ML is closed under uniform substitution, also $\varphi \notin \mathsf{ML}$. This shows that $\mathsf{ML} \subseteq \mathrm{Sch}(\mathsf{InqL})$.
For the converse inclusion, suppose $\varphi(p_1, \ldots, p_n) \notin \mathsf{ML}$. This means that there is a model $M = (F, V)$, where F is a Medvedev frame, and a point w in this model, such that $M, w \nVdash \varphi$. Now, the idea is to use lemma 4.5 to transfer this counterexample to the Kripke model M_I for inquisitive semantics. In order to do so, however, we need our starting model to be a *negative* Medvedev model. Our model is indeed a Medvedev model, but there is no reason why the valuation V should be negative. Therefore, what we want to do is replace V by a negative valuation $\widehat{V}$, and then simulate the behaviour of the propositional letters $p_1, \ldots, p_n$ with complex formulae $\psi_1, \ldots, \psi_n$.

In order to do this, associate any point u in M with a propositional letter q_u and define a new valuation $\widehat{V}$ as follows: for any point v and any proposition letter q_u, take $v \in \widehat{V}(q_u)$ if and only if $v \subseteq u$. For proposition letters q which are not of the shape q_u for some u, take $\widehat{V}(q) = \emptyset$. Then define $\widehat{M} = (F, \widehat{V})$.

Notice that the valuation $\widehat{V}$ is indeed negative. For, take any letter q_u and suppose that a certain point v is not in $\widehat{V}(q_u)$: then $v \not\subseteq u$, so we can take an element $x \in v - u$. Since $\{x\} \not\subseteq u$, $\{x\} \notin \widehat{V}(q_u)$, and therefore, since singletons are endpoints and thus behave classically, we have $\widehat{M}, \{x\} \Vdash \neg q_u$. Finally, since $\{x\} \subseteq v$, $\{x\}$ is a successor of v, and therefore $\widehat{M}, v \nVdash \neg\neg q_u$. So indeed $\widehat{M}$ is a negative Medvedev model, and lemma 4.5 applies, yielding a p-morphism $\eta : \widehat{M} \to M_I$.
We now turn to the second task, namely, find a complex formula ψ_i that simulates in $\widehat{M}$ the behaviour of the atom p_i in M. For $1 \leq i \leq n$, define $\psi_i := \bigvee_{v \in V(p_i)} q_v$. We are going to show that for any point u:

$$M, u \Vdash p_i \iff \widehat{M}, u \Vdash \psi_i$$

If $M, u \Vdash p_i$, i.e. if $u \in V(p_i)$, then since $\widehat{M}, u \Vdash q_u$ we immediately have that $\widehat{M}, u \Vdash \bigvee_{v \in V(p_i)} q_v$. That is, $\widehat{M}, u \Vdash \psi_i$.
Conversely, if $\widehat{M}, u \Vdash \psi_i$, then there is a point $v \in V(p_i)$ such that $u \in \widehat{V}(q_v)$, which in turn, by definition of $\widehat{V}$, means that $u \subseteq v$. But then, by persistence, $u \in V(p_i)$, that is, $M, u \Vdash p_i$. This proves the above equivalence. Now, it follows immediately that for any point u:

$$M, u \Vdash \varphi(p_1, \ldots, p_n) \iff \widehat{M}, u \Vdash \varphi(\psi_1, \ldots, \psi_n)$$

In particular, $\widehat{M}, w \nVdash \varphi(\psi_1, \ldots, \psi_n)$. Thus, using the p-morphism $\eta : \widehat{M} \to M_I$ provided by lemma 4.5 we finally get that $M_I, \eta(w) \nVdash \varphi(\psi_1, \ldots, \psi_n)$. Therefore, $\varphi(\psi_1, \ldots, \psi_n) \notin \mathsf{InqL}$ and thus $\varphi(p_1, \ldots, p_n) \notin \mathrm{Sch}(\mathsf{InqL})$. □

Observe that the given proof in fact establishes something stronger than the equality $\mathrm{Sch}(\mathsf{InqL}) = \mathsf{ML}$. It shows that in order to falsify a formula $\varphi \notin \mathrm{Sch}(\mathsf{InqL})$ we do not have to look at arbitrary substitution instances of φ; it suffices to take into consideration substitutions of atomic proposition letters with arbitrarily large disjunctions of atoms. This yields the following corollary.

Corollary 4.6. For any formula $\varphi(p_1, \ldots, p_n)$, the following are equivalent:

1. $\varphi(p_1, \ldots, p_n) \in \mathsf{ML}$;
2. $\varphi(\bigvee_{1 \leq i \leq k} p_1^i, \ldots, \bigvee_{1 \leq i \leq k} p_n^i) \in \mathsf{InqL}$ for all $k \in \omega$;
3. $\varphi(\bigvee_{1 \leq i \leq k} \neg p_1^i, \ldots, \bigvee_{1 \leq i \leq k} \neg p_n^i) \in \Lambda$ for all $k \in \omega$, where Λ is any intermediate logic with $\Lambda^n = \mathsf{InqL}$.

4.1 Some Notes on Medvedev's Logic

Medvedev's logic arises from interpreting propositional formulas as finite problems. This *finite problem semantics* was introduced in (Medvedev, 1962). In (Medvedev, 1966), it was shown that the associated logic can be characterized in terms of Kripke models as the logic of the class **Med**. The quest for an axiomatization of ML did not produce significant results until Maksimova *et al.* (1979) proved that ML is not finitely axiomatizable and indeed not axiomatizable with a finite number of propositional letters. The question of whether ML admits a recursive axiomatization (equivalently, of whether ML is decidable) is a long-standing open problem.

This makes the results we just established particularly interesting. For, in the first place we have seen that $\mathsf{ML} = \mathrm{Sch}(\mathsf{InqL}) = \mathrm{Sch}(\mathsf{KP}^n) = \mathrm{Sch}(\mathsf{ND}^n)$, which means that the systems KP^n and ND^n give pseudo-axiomatizations of ML: they derive 'slightly'

more formulas than those in ML, but if we restrict our attention to the schematic validities, then we have precisely Medvedev's logic.

In the second place, corollary 4.6 provides a connection between Medvedev's logic and other intermediate logics, among which the well-understood Kreisel-Putnam logic, that might pave the way for new attempts to solve the decidability problem for ML.

For instance—since both InqL and KP are decidable—if it were possible to find a finite bound b for the maximum number k of disjuncts that we need to use in order to falsify a non-schematically valid formula φ (possibly depending on the number of propositional letters in φ) then ML would be decidable. For, to determine whether $\varphi \in \mathsf{ML}$ it would then suffice to check whether the formula $\varphi(\bigvee_{1\leq i\leq k} \neg p_1^i, \ldots, \bigvee_{1\leq i\leq k} \neg p_n^i)$ is in KP for all $k \leq b$, and this procedure can be performed in a finite amount of time.

It is possible to show (see Ciardelli, 2009) that for formulas containing a single proposition letter p, such a bound exists and equals 2. As a consequence, the one-letter fragment of ML is decidable (and in fact coincides with the well-known Scott logic). This was already known, but the argument is new and could perhaps be generalized.

5 The Range of Intermediate Logics whose Negative Variant is InqL

Using the result that the schematic fragment of InqL coincides with ML it is possible to strengthen the completeness result obtained in section 3.6: we can give a complete characterization of the range of intermediate logics whose negative variant coincides with InqL.

Theorem 5.1 (Range of intermediate logics whose negative variant is InqL)**.** For any intermediate logic Λ:

$$\Lambda^n = \mathsf{InqL} \quad \Longleftrightarrow \quad \mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$$

We articulate the proof of this theorem in two lemmata.

Lemma 5.2. For any intermediate logic Λ, if $\Lambda^n = \mathsf{InqL}$, then $\mathsf{ND} \subseteq \Lambda$.

Proof. By contraposition, suppose $\mathsf{ND} \not\subseteq \Lambda$. Then there is a number k for which the formula $\mathtt{ND}_k := \big(\neg p \to \bigvee_{1\leq i\leq k} \neg q_i\big) \quad \longrightarrow \quad \bigvee_{1\leq i\leq k} (\neg p \to \neg q_i)$ is not in Λ. Note that this formula is nothing but φ^n where φ denotes the formula

$$\Big(p \to \bigvee_{1\leq i\leq k} q_i\Big) \quad \longrightarrow \quad \bigvee_{1\leq i\leq k} (p \to q_i)$$

But then φ^{nn} cannot be in Λ. For if it were—Λ being closed under uniform substitution—φ^{nnn} should also be in Λ, and so should the equivalent formula φ^n. But φ^n is *not* in Λ. Thus, $\varphi^{nn} \notin \Lambda$, whence $\varphi^n \notin \Lambda^n$. On the other hand, $\varphi^n = \mathtt{ND}_k \in \mathsf{InqL}$. So, $\Lambda^n \neq \mathsf{InqL}$. □

Lemma 5.3. For any intermediate logic Λ, if $\Lambda^n = \mathsf{InqL}$, then $\Lambda \subseteq \mathsf{ML}$.

Proof. We know that Λ is always included in Λ^n, so if $\Lambda^n = \mathsf{InqL}$ we have $\Lambda = \mathrm{Sch}(\Lambda) \subseteq \mathrm{Sch}(\Lambda^n) = \mathrm{Sch}(\mathsf{InqL}) = \mathsf{ML}$, where the first equality uses the fact that Λ is closed under uniform substitution. □

Proof of theorem 5.1. Let Λ be an intermediate logic. The operation of negative variant is obviously monotone, so if $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$ we have $\mathsf{InqL} = \mathsf{ND}^n \subseteq \Lambda^n \subseteq \mathsf{ML}^n = \mathsf{InqL}$, where the first and the last equality come from corollary 3.34.

On the other hand, the previous two lemmata together show that $\Lambda^n = \mathsf{InqL}$ implies $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$. This completes the proof of the theorem. □

Note that as a corollary of this theorem we can easily derive a well-known result due to Maksimova (1986), which was already mentioned at the end of section 3.6.

Corollary 5.4. If $\Lambda \supseteq \mathsf{ND}$ is a logic with the disjunction property, then $\Lambda \subseteq \mathsf{ML}$. In particular, ML is a *maximal* logic with the disjunction property.

Proof. According to theorem 3.33, if $\Lambda \supseteq \mathsf{ND}$ is a logic with the disjunction property, then $\Lambda^n = \mathsf{InqL}$ and thus, by theorem 5.1, $\Lambda \subseteq \mathsf{ML}$. □

Finally, theorem 5.1 can easily be extended to a *strong* completeness result.

Theorem 5.5 (Strong Completeness)**.** For any intermediate logic Λ, the following two conditions are equivalent:

1. $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$
2. For any set of formulas Θ and any formula φ: $\quad \Theta \models_{\mathsf{InqL}} \varphi \iff \Theta \vdash_{\Lambda^n} \varphi$

Proof. First let Λ be an intermediate logic that satisfies condition 2. Then, in particular, $\mathsf{InqL} = \Lambda^n$, and therefore, by theorem 5.1, $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$.

Now let us assume that Λ is an intermediate logic that satisfies condition 1, and show that $\vdash_{\Lambda^n}$ is sound and complete with respect to $\models_{\mathsf{InqL}}$. The soundness direction is straightforward since $\Lambda \subseteq \mathsf{InqL}$ (for, $\Lambda \subseteq \Lambda^n$ and $\Lambda^n = \mathsf{InqL}$), $\neg\neg p \to p \in \mathsf{InqL}$, and the set of formulas supported by a state is closed under modus ponens. For the completeness direction, suppose that $\Theta \models_{\mathsf{InqL}} \varphi$. Then, by compactness (theorem 3.10), there are formulas $\theta_1, \ldots, \theta_k \in \Theta$ such that $\theta_1, \ldots, \theta_k \models_{\mathsf{InqL}} \varphi$, which by the deduction theorem amounts to $\theta_1 \wedge \cdots \wedge \theta_k \to \varphi \in \mathsf{InqL}$. Then by theorem 5.1, $\theta_1 \wedge \cdots \wedge \theta_k \to \varphi \in \Lambda^n$, whence clearly $\Theta \vdash_{\Lambda^n} \varphi$. □

6 InqL as the disjunctive-negative fragment of IPL

As we already observed, the meanings of inquisitive semantics are sets of alternatives, where each alternative is a classical meaning. This essential feature of the semantics is mirrored on the syntactic, logical level by the fact that any formula φ is equivalent to a disjunction of negations $\textsc{dnt}(\varphi)$.

The completeness result in section 3 was based on the insight that preservation of logical equivalence under $\textsc{dnt}$ is an essential feature of the logic InqL. But there is even more to say about $\textsc{dnt}$: in this section we will show that it constitutes a *translation* of InqL into IPL, in the following sense (cf. Epstein *et al.*, 1995, chapter 10):

Definition 6.1 (Translations between logics)**.** Let L, L' be two logics arising from two entailment relations $\models_L$ and $\models_{L'}$. We say that a mapping t from formulas in the language of L to formulas in the language of L' is a *translation* from L to L' in case for any set of formulas Θ and any formula φ we have:

$$\Theta \models_L \varphi \iff t[\Theta] \models_{L'} t(\varphi)$$

where $t[\Theta] = \{t(\theta) \mid \theta \in \Theta\}$.

Moreover, we will show that the disjunctive-negative fragment of InqL coincides with the one of IPL, and that InqL is in fact isomorphic to the disjunctive-negative fragment of IPL through the translation $\textsc{dnt}$ (just as CPL is isomorphic to the negative fragment of IPLthrough the translation mapping φ to $\neg\neg\varphi$).

Let us call a formula *disjunctive-negative* in case it is a disjunction of negations. The following proposition says that inquisitive entailment and intuitionistic entailment agree as far as disjunctive-negative formulas are concerned.

Proposition 6.2. If φ is a disjunctive-negative formula and Θ a set of disjunctive-negative formulas, then: $\Theta \models_{\mathsf{InqL}} \varphi \iff \Theta \models_{\mathsf{IPL}} \varphi$.

Proof. Consider an arbitrary set Θ of disjunctive-negative formulas and a disjunctive-negative formula $\varphi = \neg\xi_1 \vee \cdots \vee \neg\xi_k$. If $\Theta \models_{\mathsf{InqL}} \varphi$, then by compactness and the deduction theorem there must be $\theta_1, \ldots, \theta_n \in \Theta$ such that $\theta_1 \wedge \cdots \wedge \theta_n \to \varphi \in \mathsf{InqL}$.

Now since each θ_k is a disjunction of negations and since the distributive laws hold in intuitionistic logic, in IPL we can turn $\theta_1 \wedge \cdots \wedge \theta_n$ into a disjunction of conjunctions of negations. In turn, a conjunction of negations is equivalent to a negation in intuitionistic logic. So we can find formulas $\chi_1, \ldots, \chi_m$ such that $\theta_1 \wedge \cdots \wedge \theta_n \equiv_{\mathsf{IPL}} \neg\chi_1 \vee \cdots \vee \neg\chi_m$. But then $(\theta_1 \wedge \cdots \wedge \theta_n \to \varphi) \equiv_{\mathsf{IPL}} (\neg\chi_1 \vee \cdots \vee \neg\chi_m \to \varphi) \equiv_{\mathsf{IPL}} \bigwedge_{1 \leq i \leq m} (\neg\chi_i \to \varphi)$.

Equivalence in IPL implies equivalence in InqL, so $\theta_1 \wedge \cdots \wedge \theta_n \to \varphi \in \mathsf{InqL}$ implies that $\bigwedge_{1 \leq i \leq m} (\neg\chi_i \to \varphi) \in \mathsf{InqL}$, which in turn means that for each $1 \leq i \leq n$ we have $\neg\chi_i \to \varphi \in \mathsf{InqL}$.
Writing out φ, this amounts to $\neg\chi_i \to \neg\xi_1 \vee \cdots \vee \neg\xi_k \in \mathsf{InqL}$. But since InqL contains the Kreisel-Putnam axiom, it follows that $\bigvee_{1 \leq j \leq k} (\neg\chi_i \to \neg\xi_j) \in \mathsf{InqL}$, and therefore, as InqL has the disjunction property, for some $1 \leq j \leq k$ we must have that $\neg\chi_i \to \neg\xi_j \in \mathsf{InqL} \subseteq \mathsf{CPL}$.

Now, $\neg\chi_i \to \neg\xi_j \equiv_{\mathsf{IPL}} \neg\neg(\neg\chi_i \to \neg\xi_j)$, and since CPL and IPL agree about negations (Chagrov and Zakharyaschev, 1997, corollary 2.49), also $\neg\chi_i \to \neg\xi_j \in \mathsf{IPL}$, whence *a fortiori* $\neg\chi_i \to \varphi \in \mathsf{IPL}$.
But since this can be concluded for each i, we have $\bigwedge_{1 \leq i \leq m} (\neg\chi_i \to \varphi) \in \mathsf{IPL}$, and therefore also the equivalent formula $\theta_1 \wedge \cdots \wedge \theta_n \to \varphi$ must be in IPL. But then obviously $\Theta \models_{\mathsf{IPL}} \varphi$.
The converse implication is trivial, as InqL extends IPL. □

As a particular case of this proposition, let us remark that for any disjunctive-negative formula φ we have $\varphi \in \mathsf{InqL} \iff \varphi \in \mathsf{IPL}$.

Corollary 6.3. $\textsc{dnt}$ is a translation of InqL into IPL.

Proof. We have to show that for any Θ and any φ:

$$\Theta \models_{\mathsf{InqL}} \varphi \iff \textsc{dnt}[\Theta] \models_{\mathsf{IPL}} \textsc{dnt}(\varphi)$$

where $\textsc{dnt}[\Theta] = \{\textsc{dnt}(\theta) \mid \theta \in \Theta\}$. It follows from proposition 3.13 that $\Theta \models_{\mathsf{InqL}} \varphi \iff \textsc{dnt}[\Theta] \models_{\mathsf{InqL}} \textsc{dnt}(\varphi)$. But $\textsc{dnt}(\psi)$ is always a disjunctive-negative formula. So, by proposition 6.2, $\textsc{dnt}[\Theta] \models_{\mathsf{InqL}} \textsc{dnt}(\varphi) \iff \textsc{dnt}[\Theta] \models_{\mathsf{IPL}} \textsc{dnt}(\varphi)$ and we are done. □

Observe that if the map t is a translation from a logic L to another logic L', then t naturally lifts to an embedding $\bar{t} : \mathcal{L}/_{\equiv_L} \to \mathcal{L}/_{\equiv_{L'}}$ of the Lindenbaum-Tarski algebra of L into the Lindenbaum-Tarski algebra of L', given by $\bar{t}([\psi]_{\equiv_L}) := [t(\psi)]_{\equiv_{L'}}$.[3]

[3] For more details on the issues of translations between logics, see (Epstein *et al.*, 1995).

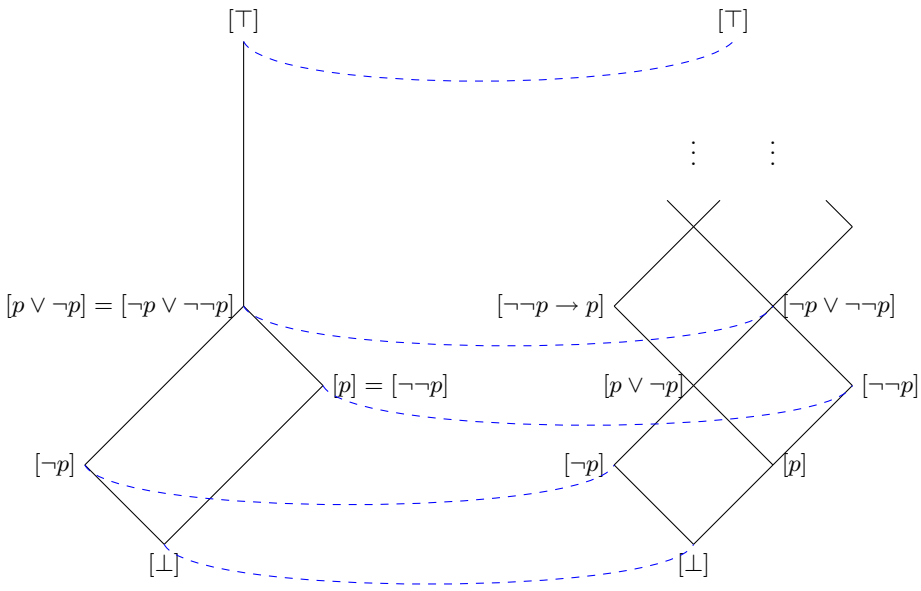


Fig. 3 Embedding of the Lindenbaum-Tarski algebra of InqL (on the left), into the Lindenbaum-Tarski algebra of IPL (the Rieger-Nishimura lattice, on the right), for the singleton set of proposition letters $\mathcal{P} = \{p\}$.

Since we have seen that DNT is a translation from InqL to IPL, the map $\overline{\text{DNT}}$ defined by $\overline{\text{DNT}}([\psi]_{\equiv_{\mathsf{InqL}}}) = [\text{DNT}(\psi)]_{\equiv_{\mathsf{IPL}}}$ is an embedding of the Lindenbaum-Tarski algebra of InqL into the one of IPL. For the singleton set of propositional letters, this embedding is depicted in figure 6.

Now, for any ψ, $\text{DNT}(\psi)$ is a disjunctive-negative formula. Conversely, consider a disjunctive-negative formula ψ. Since $\psi \equiv_{\mathsf{InqL}} \text{DNT}(\psi)$ but both ψ and $\text{DNT}(\psi)$ are disjunctive-negative, it follows from proposition 6.2 that $\psi \equiv_{\mathsf{IPL}} \text{DNT}(\psi)$; in other words, we have $[\psi]_{\equiv_{\mathsf{IPL}}} = [\text{DNT}(\psi)]_{\equiv_{\mathsf{IPL}}} = \overline{\text{DNT}}([\psi]_{\equiv_{\mathsf{IPL}}})$, so $[\psi]_{\equiv_{\mathsf{IPL}}}$ is in the image of the embedding DNT.

This shows that the image of the embedding DNT is precisely the set of equivalence classes of disjunctive-negative formulas. In other words, just like CPL is isomorphic to the negative fragment of IPL, for InqL we have the following result.

Proposition 6.4. InqL is isomorphic to the disjunctive-negative fragment of IPL.

As a corollary of the well-known fact that CPL is isomorphic to the negative fragment of IPL we know that, for any n, there are exactly 2^{n+1} intuitionistically non-equivalent negative formulas in $\mathcal{L}_{\{p_1,\ldots,p_n\}}$, just as many as there are classically non-equivalent formulas in $\mathcal{L}_{\{p_1,\ldots,p_n\}}$.

Analogously, our result that InqL is isomorphic to the disjunctive-negative fragment of IPL comes with the corollary that there are exactly as many intuitionistically non-equivalent disjunctive-negative formulas in $\mathcal{L}_{\{p_1,\ldots,p_n\}}$ as there are inquisitively non-equivalent formulas in $\mathcal{L}_{\{p_1,\ldots,p_n\}}$.

The number of inquisitively non-equivalent formulas in $\mathcal{L}_{\{p_1,\ldots,p_n\}}$ is given by the number of distinct inquisitive meanings built up from indices in $\mathcal{I}_{\{p_1,\ldots,p_n\}}$. Such inquisitive meanings are nothing but antichains of the powerset algebra $\wp(\mathcal{I}_{\{p_1,\ldots,p_n\}})$.

This algebra is isomorphic to $\wp(2^n)$, since $\mathcal{I}_{\{p_1,\ldots,p_n\}} = \wp(\{p_1,\ldots,p_n\})$ contains 2^n elements. Therefore, letting $D(n)$ denote the number of antichains of the powerset algebra $\wp(n)$, we have the following fact.

Corollary 6.5. For any n, there are exactly $D(2^n)$ intuitionistically non-equivalent disjunctive-negative formulas in $\mathcal{L}_{\{p_1,\ldots,p_n\}}$.

The numbers $D(n)$ are known as Dedekind numbers, and although no simple formula is known for their calculation, their values for small n have been computed and are available online, see for instance: www.research.att.com/~njas/sequences/A014466.

The number of inquisitive meanings in one propositional letter is 5, as displayed by the above picture; with two letters we have 167 meanings, and with three the number leaps to 56130437228687557907787.

7 The Inquisitive Hierarchy

Inquisitive semantics was first developed by Groenendijk (2008c) and Mascarenhas (2008). The original formulation of the semantics was different from the one considered here: formulas were evaluated w.r.t. *ordered pairs* of indices rather than w.r.t. arbitrary *sets* of indices. The intuition underlying that semantics came from the tradition of Groenendijk's logic of interrogation (Groenendijk, 1999; ten Cate and Shan, 2007), where a formula disconnects two indices in the common ground in case it expresses an interest in the difference between them. We will refer to that system as the *pair semantics*, and to the system considered in the present paper as the *generalized semantics*.

Going through the clauses that define the pair semantics it is immediately clear that for any indices v, w and any formula φ, $\langle v, w\rangle \models \varphi$ amounts precisely to our $\{v, w\} \models \varphi$ (that is, the *order* of the pair $\langle v, w\rangle$ is irrelevant). In this sense the pair semantics can be seen as a fragment of the generalized semantics, namely the fragment dealing only with states of size at most two. This restriction on the cardinality of states gives rise to a logic InqL_2, much stronger than InqL, which has been studied and axiomatized by Mascarenhas (2009).

In general, we may consider the operation of restricting our semantics by allowing only states of cardinality at most n. As we shall see, doing so gives rise to a hierarchy of strictly shrinking logics whose limit is InqL.

Remark 7.1. Throughout this section we assume a countably infinite set of propositional letters $\mathcal{P} = \{p_i \mid i \in \omega\}$.

Definition 7.2 (The Inquisitive Hierarchy)**.**
For $k \in \omega$, define $\mathsf{InqL}_k = \{\varphi \mid s \models \varphi$ for any state s with $|s| \leq k\}$.

The only state of cardinality zero is $\emptyset$, which supports any formula, so InqL_0 is the inconsistent logic. Moreover, since singleton states behave like indices, InqL_1 is classical logic. And, clearly, InqL_2 is the logic arising from the pair semantics. The following fact is a trivial consequence of the definition of the hierarchy.

Remark 7.3. $\mathsf{InqL} = \bigcap_{k\in\omega} \mathsf{InqL}_k$

We will now define, for each $k \in \omega$, a formula δ_k that characterizes the class of intuitionistic Kripke frames of depth at most k. That is, for each $k \in \omega$, δ_k is valid on an intuitionistic Kripke frame F if and only if the depth of F is at most k.

We will then show, first, that for any $k \in \omega$, δ_k is in InqL_k but not in InqL_{k+1} (which means that the inquisitive hierarchy is a sequence of strictly shrinking logics), and second, that for any $k \in \omega$, adding all substitution instances of δ_k to an axiomatization of InqL yields an axiomatization of InqL_k.

Definition 7.4. The formulas δ_k, $k \in \omega$ are defined inductively as follows.

- $\delta_0 := \bot$
- $\delta_{k+1} := p_{k+1} \vee (p_{k+1} \to \delta_k)$

Proposition 7.5. For any natural k, $\delta_k \in \mathsf{InqL}_k$ but $\delta_k \notin \mathsf{InqL}_{k+1}$.

Proof. First let us remark that for any finite non-empty state s, the depth of the subframe $(F_I)_s$ generated by the point s in the frame F_I of the Kripke model for inquisitive logic is equal to $|s|$. This can be checked by an easy induction.

Now, first consider any state s with $|s| \leq k$: we have to check that $s \models \delta_k$. We may assume $s \neq \emptyset$, as our claim is trivial for the empty state. Now as the depth of the frame $(F_I)_s$ is $|s| \leq k$, the formula δ_k is valid on $(F_I)_s$, whence in particular $((F_I)_s, V_I), s \Vdash \delta_k$, where V_I is the valuation of M_I. But then, since Kripke satisfaction is invariant under taking generated submodels we have $M_I, s \Vdash \delta_k$, which by proposition 3.18 amounts to $s \models \delta_k$. This shows that δ_k is supported by any state of size at most k. Hence, $\delta_k \in \mathsf{InqL}_k$.

Second, we have to show that $s \not\models \delta_k$ for some state s with $|s| = k+1$. We shall proceed by induction on k. For $k = 0$, simply take $s_0 = \{w\}$ where w is the index making all proposition letters true.

Now, inductively, we can assume that we have a state s_k such that $|s_k| = k+1$, $|s_k| \not\models \delta_k$, and *moreover* all indices in s_k make p_j true for all $j \geq k+1$. Now simply let $s_{k+1} := s_k \cup \{w\}$ where w is the index making a letter p_j true iff $j \geq k+2$. It is immediate to check that $s_{k+1} \not\models \delta_{k+1}$, and since $|s_{k+1}| = k+2$ and all indices in s_{k+1} make p_j true for all $j \geq k+2$, the inductive step is complete. □

This proves that the inquisitive hierarchy is indeed a hierarchy of strictly shrinking logics: for any $k \in \omega$, $\mathsf{InqL}_k \supsetneq \mathsf{InqL}_{k+1}$. We now turn to the axiomatization of the logics in the hierarchy.

Definition 7.6 (Δ_k)**.** For any natural k we denote by Δ_k the intermediate logic axiomatized by the formula δ_k.

Theorem 7.7 (Axiomatization of the inquisitive hierarchy)**.** Let Λ be an intermediate logic with $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$. Then $\mathsf{InqL}_k = (\Lambda + \Delta_k)^n$.

In other words, the theorem states that InqL_k is soundly and completely axiomatized by a derivation system having modus ponens as derivation rule, and the following axioms:

1. Λ, or axioms for Λ in schematic form;
2. Δ_k, or δ_k as an axiom scheme;
3. $\neg\neg p \to p$ for all $p \in \mathcal{P}$.

For instance, if we choose $\Lambda = \mathsf{KP}$, then the theorem says that we can take our derivation system to have, as axioms, all substitution instances of the formulas $\mathtt{KP}$ and δ_k, plus atomic double negation axioms.

Proof. For the soundness direction, since the set InqL_k is closed under modus ponens and contains Λ and the atomic double negation axioms, it suffices to check that any substitution instance ${\delta_k}^*$ of δ_k is in InqL_k. Consider a state s with $0 < |s| \leq k$ (the case $|s| = 0$ is trivial): then the generated subframe $(F_I)_s$ of s in the frame F_I has depth at most k, so δ_k is valid on $(F_I)_s$, and since the logic of a frame is closed under uniform substitution, ${\delta_k}^*$ is valid on $(F_I)_s$ as well. In particular we have $((F_I)_s, V_I), s \Vdash {\delta_k}^*$, which by invariance of Kripke-satisfaction under generated submodels yields $M_I, s \Vdash {\delta_k}^*$ and thus $s \models {\delta_k}^*$.

For completeness, suppose $\varphi \notin (\Lambda + \Delta_k)^n$, that is, assume that $\Delta_k \nvdash_{\Lambda^n} \varphi$. By the strong completeness of Λ^n (theorem 5.5) there must be a state s such that $s \models \Delta_k$ but $s \not\models \varphi$. Now in order to conclude that $\varphi \notin \mathsf{InqL}_k$ it suffices to show that $s \models \Delta_k$ implies $|s| \leq k$. This is the content of the following lemma. □

Lemma 7.8. For any $k \in \omega$, if $|s| > k$ then there is a substitution instance ${\delta_k}^*$ of the formula δ_k such that $s \not\models {\delta_k}^*$.

Proof. We will proceed by induction on k. The case for $k = 0$ is trivial: $\delta_0 = \bot$ is not supported by any non-empty state.

Now, assume our claim holds for a number k and consider a state s with $|s| > k+1$. We may assume that s is finite: if not, just replace it by a finite substate $s' \subseteq s$ with $|s'| > k+1$: then by persistence, once we find an instance of δ_k which is not supported by s', this cannot be supported by s either.

Fix an index $w \in s$: exploiting the fact that w must differ from any other $w' \in s$ on some letter and that s contains only finitely many indices, we can easily find a formula γ such that $w \models \gamma$ but $w' \not\models \gamma$ for any $w' \neq w$ in s.

Now, since $|s - \{w\}| > k$, by induction hypothesis there is a substitution instance ${\delta_k}^*$ of δ_k such that $s - \{w\} \not\models \delta_k$. Since δ_k contains only the variables $p_1, \ldots, p_k$, the substitution we need in order to get ${\delta_k}^*$ from δ_k does not concern the variable p_{k+1}, for which we are free to choose a substitute: thus, the formula ${\delta_{k+1}}^* := \neg\gamma \vee (\neg\gamma \to {\delta_k}^*)$ is indeed a substitution instance of $\delta_{k+1} = p_{k+1} \vee (p_{k+1} \to \delta_k)$.

Now $s \not\models \neg\gamma$, because $w \in s$ and $w \not\models \neg\gamma$; on the other hand, $s \not\models \neg\gamma \to {\delta_k}^*$, because $s - \{w\} \not\models \delta_k$ while $s - \{w\} \models \neg\gamma$ by proposition 2.6. Thus, $s \not\models {\delta_{k+1}}^*$ and we are done: we have shown that for any state s with $|s| > k+1$ there is a substitution instance of δ_{k+1} which is not supported by s. □

Note that, proceeding in exactly the same way, we could in fact have shown that, for any Λ such that $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$, $(\Lambda + \Delta_k)^n$ provides a *strongly* complete axiomatization of InqL_k, in the sense that it captures the notion of entailment $\models_{\mathsf{InqL}_k}$ that results from restricting inquisitive semantics to states of size at most k. That is, for any set Θ and any formula φ,

$$\Theta \models_{\mathsf{InqL}_k} \varphi \iff \Theta \vdash_{(\Lambda+\Delta_k)^n} \varphi$$

8 A Plea for the Generalized Semantics

We have seen that in the pair semantics, formulas are evaluated with respect to ordered pairs of indices. The possibilities for a formula φ are then defined as maximal states such that any pair of indices in that state supports φ. This notion of possibilities subtly differs from the one that the generalized semantics gives rise to. Thus, the pair semantics and the generalized semantics yield a different notion of meaning. In this

section, we compare these two notions of meaning and argue that the differences speak in favour of the generalized semantics.

In order to make such a judgment, we must first of all determine a suitable criterion for comparison. In order to do so, we return to one of the main sources of inspiration behind inquisitive semantics, the 'Gricean picture of disjunction' (Grice, 1989, p. 68):

> A standard (if not the standard) employment of 'or' is in the specification of possibilities (one of which is supposed by the speaker to be realized, although he does not know which one), each of which is relevant in the same way to a given topic. 'A or B' is characteristically employed to give a partial answer to some [wh]-question, to which each disjunct, if assertible, would give a fuller, more specific, more satisfactory answer.

This picture has played an important role in the development of inquisitive semantics (cf. Groenendijk, 2008b), and indeed, a disjunction $p \vee q$ is assigned a meaning consisting of two alternative possibilities, $|p|$ and $|q|$.

Now, the Gricean picture of disjunction is of course not intended to apply only to disjunctions with *two* disjuncts. It applies just as well to disjunctions with three or more disjuncts. For instance, the idea is that a triple disjunction $p \vee q \vee r$ is used to specify three possibilities, $|p|$, $|q|$ and $|r|$. One criterion, then, for comparing different implementations of inquisitive semantics, is that the Gricean picture of disjunction should be captured for disjunctions of arbitrary length.

This is indeed what the generalized semantics does, unlike any other element of the inquisitive hierarchy. Let us consider the pair semantics in particular. This semantics assigns to $p \vee q \vee r$ the three possibilities $|p|$, $|q|$ and $|r|$, but also an additional possibility $t = \{111, 110, 101, 011\}$ (since every *pair* of indices in t supports $p \vee q \vee r$).

More generally, a disjunction $p_1 \vee \ldots \vee p_{n+1}$ will be problematic for any element of the inquisitive hierarchy that looks only at states of size at most n. The Gricean picture is only captured in full generality if states of arbitrary size are taken into account.

Now let us consider another criterion, which concerns the intuition that the possibilities for a sentence φ correspond to the alternative ways in which φ may be resolved. Again, upon close examination, it turns out that this intuition is captured by the generalized semantics, but not by any other element of the inquisitive hierarchy.

In section 2.4 we saw that a state s supports a formula φ precisely in case the issue raised by φ is settled in s. Thus, possibilities in the generalized semantics are maximal states in which the issue raised by a formula is settled in a particular way; in this sense, the possibilities that we get in the generalized semantics embody the possible ways of resolving the issue raised by the formula.

It is easy to see that the pair semantics can only disagree with the generalized semantics about the meaning of a formula φ in case one of the possibilities that the pair semantics assigns to φ, call it t, does in fact not support φ. In this case, according to proposition 2.26, φ must be informative or inquisitive in t; but it cannot be informative, since both semantics yield the same, classical treatment of information, so it must be inquisitive: for instance, $p \vee q \vee r$ is inquisitive in the state $\{111, 110, 101, 011\}$, which is a possibility according to the pair semantics.

This means that the issue raised by φ is still open in t: t does not correspond to a possible way of resolving φ. Thus, the pair semantics does not yield the intended notion of possibilities, and the same point can easily be made for any other element of the inquisitive hierarchy.

9 Support as 'Knowing How'

The motivation for inquisitive semantics lies in certain conceptual ideas about information exchange through conversation. The link between these ideas and the formal semantics is usually established at the level of propositions. However, propositions are defined indirectly, through the notion of support. In this final section we suggest an intuitive interpretation of this basic notion of support, which allows for a new perspective on the semantics and the associated logic, and in particular on some of the observations made in previous sections.

We have already seen that states in inquisitive semantics can be conceived of as information states. Traditionally, an information state s is taken to support a formula φ iff it it is known in s *that* φ *is true*. This is *not* how support should be thought of in the present setting. However, there is a closely related interpretation that *is* appropriate: '$s \models \varphi$' can be read as stating the conditions under which it is known in s *how* φ *is realized*.

Under this perspective, the basic clause in the definition of support states that atoms can only be realized in *one* way: the fact that they name must obtain. This special character of atoms explains the fact that inquisitive logic is not closed under uniform substitution: atoms are not placeholders denoting arbitrary meanings, but really represent 'atomic' meanings lacking any inquisitive complexity. They are simply names for facts, states of affairs that may or may not obtain in the world. This guarantees that each of the alternatives proposed by a formula is in fact expressible and can be selected by the dialogue participants through some utterance. This desirable feature of the system that would be lost if atoms were allowed to be inquisitive.

Returning to the definition of support, the recursive clauses can be read as follows: one never knows 'how $\bot$' unless one's information state is inconsistent; knowing 'how $\varphi \vee \psi$' requires knowing 'how φ' or 'how ψ'; and knowing 'how $\varphi \wedge \psi$' requires knowing 'how φ' and 'how ψ'. Finally, the support-clause for implication becomes particularly perspicuous under this interpretation. A state s supports $\varphi \to \psi$ iff every substate of s that supports φ also supports ψ. That is, we know how $\varphi \to \psi$ is realized iff in every future information state where we know how φ is realized, we also know how ψ is realized. Thus, knowing 'how $\varphi \to \psi$' requires knowing not only *that* ψ is realized whenever φ is, but also *how* the way ψ is realized depends on the way φ is realized. In other words, what is required is a method for turning knowledge as to how φ is realized into knowledge as to how ψ is realized, just like in intuitionistic logic having a proof for $\varphi \to \psi$ amounts to having a method for turning a proof for φ into a proof for ψ.

These observations naturally suggest a formal, inductive definition of the 'ways in which a formula can be realized'. There is only one subtlety that we have to take care of: in order to avoid multiple copies of the same realization, we fix a normal form for formulas in classical logic such that the normal form of each formula is disjunction-free. Denote the normal form of a formula φ by φ_{nf}. We may assume that $\bot_{nf} = \bot$ and that $p_{nf} = p$ for proposition letters. We can then define *realizations* as follows.

Definition 9.1 (Realizations)**.**

1. $\mathcal{R}(p) = \{p\}$ for $p \in \mathcal{P}$
2. $\mathcal{R}(\bot) = \{\bot\}$
3. $\mathcal{R}(\varphi \vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$
4. $\mathcal{R}(\varphi \wedge \psi) = \{(\rho \wedge \sigma)_{nf} \mid \rho \in \mathcal{R}(\varphi) \text{ and } \sigma \in \mathcal{R}(\psi)\}$
5. $\mathcal{R}(\varphi \to \psi) = \{(\bigwedge_{\rho \in \mathcal{R}(\varphi)} (\rho \to f(\rho)))_{nf} \mid f : \mathcal{R}(\varphi) \to \mathcal{R}(\psi)\}$

We shall refer to the elements of $\mathcal{R}(\varphi)$ as the *realizations* of φ. It is immediate to check inductively that realizations are always disjunction-free, and therefore, by corollary 2.19, they are assertions. The following proposition—whose straightforward inductive proof is omitted—shows that possibilities mirror the alternative ways in which a formula may be realized (where 'alternative' means 'mutually incomparable with respect to entailment').

Proposition 9.2. For any formula φ, the possibilities for φ coincide with the maximal elements of $\{|\rho| \mid \rho \in \mathcal{R}(\varphi)\}$.

In terms of realizations, the intuitive interpretation of support suggested above can be made more precise: a formula φ is supported just in case it is known how φ is realized, that is, just in case a realization of φ is known.

Proposition 9.3. For any state s and formula φ,

$$s \models \varphi \iff s \subseteq |\rho| \text{ for some } \rho \in \mathcal{R}(\varphi)$$

In the light of this interpretation, the conversational effect of an utterance can be rephrased as follows. An utterance of φ provides the information *that* φ is true and it raises the issue *how* φ is realized. Therefore, realizations may also be viewed as *resolutions* of the issue raised by φ. Proposition 9.2 then says that the possibilities for φ mirror the alternative ways in which the issue raised by φ may be resolved—which is indeed what was already claimed, informally, at the end of section 8.

10 Conclusions

We investigated a generalized version of inquisitive semantics, and the logic it gives rise to. In particular, we established that InqL coincides with the negative variant of any intermediate logic Λ such that $\mathsf{ND} \subseteq \Lambda \subseteq \mathsf{ML}$. We also showed that the schematic fragment of inquisitive logic coincides with ML, thus achieving a pseudo-axiomatization of the latter, and proved that inquisitive logic is isomorphic to the disjunctive-negative fragment of intuitionistic logic. Finally, we compared the generalized version of inquisitive semantics with the original 'pair semantics', arguing in favour of the generalized system, and we presented an intuitive interpretation of the inquisitive notion of support, hopefully illuminating some of the technical results that were obtained in earlier sections.

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