

Inquisitive Semantics

Assertions, Questions, and Hybrids

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1 Introduction

Immediate predecessor of this document is the handout of my NYU-Seminar class of February 25, which in turn was drawn from the draft of a paper, which in turn, etc. I am in the process of getting at a new version of the paper. The current document is a DRAFT in an extreme sense of the word.

My current purpose with this document is no other than to present and explain the semantics, and to some extent the logic it gives rise to. It is just intended as the introduction to the logical system as such.

Whenever I have changed or expanded things to a significant extent, I publish the new version on the inquisitive semantics website.

References, acknowledgements, etc., are still largely lacking. So let me at least mention the four most important contributors to this project in general, and this paper in particular. In alphabetical order: Kata Balogh, Ivano Ciardelli, Salvador Mascarenhas, and Floris Roelofsen.

2 A Hybrid Language

We will provide an inquisitive semantics for a hybrid language of propositional logic. We call the language hybrid, because there is only a single syntactic category of sentences, whereas in the semantics we will distinguish three distinct categories of meaningful sentences: assertions, questions, and hybrids.

The basic ingredients of the language are: the *falsum* \perp ; a non-empty and finite set of propositional variables P ; and three two-place connectives by which we can form conjunctions $(\varphi \wedge \psi)$, disjunctions $(\varphi \vee \psi)$, and conditionals $(\varphi \rightarrow \psi)$, in the usual recursive way.

We standardly add negations $\neg\varphi$ to the language, defined as $\varphi \rightarrow \perp$, and the *verum* \top , defined as $\neg\perp$. We non-standardly add the *non-inquisitive closure* $!\varphi$ of a sentence φ , defined as $\neg\neg\varphi$, and the *non-informative closure* $?\varphi$ of a sentence φ , defined as $\varphi \vee \neg\varphi$.

That we call $!\varphi$ the non-inquisitive closure of φ and $?\varphi$ its non-informative closure, will be justified by the semantics, where informativeness and inquisitiveness are the basic semantic notions.

3 Basic ingredients of the semantics

Let L be a hybrid propositional language with propositional variables P , which we call atomic sentences. A suitable *index* for L is a binary valuation v for the atomic sentences $p \in P$.

By ω we denote the set of all suitable indices for L , which we also call the logical space. Since the set of atomic sentences is finite, the logical space is finite as well. If there are n atomic sentences in L , the size of the logical space is 2^n .

An *information state* for L is a non-empty set of suitable indices. We use i, j, k as variables ranging over states. The set of states is partially ordered by the (non-empty) subset relation, we read $i \subseteq j$ as *i is a substate of j* . We refer to the \subseteq -minimal substates of ω as *final states*. So, a final state $\{v\}$ consists of a single index $v \in \omega$.

4 Inquisitive Support

In the recursive statement of the semantics we define a satisfaction relation for the sentences $\varphi \in L$, relative to an information state i for L . We read $i \models \varphi$ as *state i supports φ* .

Definition 1 (Inquisitive Propositional Semantics).

1. $i \not\models \perp$
2. $i \models p$ iff for all $v \in i$: $v(p) = 1$
3. $i \models (\varphi \wedge \psi)$ iff $i \models \varphi$ and $i \models \psi$
4. $i \models (\varphi \vee \psi)$ iff $i \models \varphi$ or $i \models \psi$
5. $i \models (\varphi \rightarrow \psi)$ iff for all $j \subseteq i$: if $j \models \varphi$, then $j \models \psi$

We can paraphrase the clauses as follows:

1. No state supports the *falsum*.
2. To be a state that supports an atomic sentence, is to be a state such that the atomic sentence holds in all the indices in that state.
3. To be a state that supports a conjunction is to be a state that supports both conjuncts.
4. To be a state that supports a disjunction is to be a state that supports at least one of the disjuncts.
5. To be a state that supports a conditional sentence, is to be a state such that all of its substates that support the antecedent, also support the consequent.

It is *compliance*, to be discussed later, rather than entailment, that is the logical notion that comes with the semantics.¹ However, if only for comparison with other logics, we define entailment as well. The definition has a standard form:

¹ I am working on a new idea for defining a single notion of compliance, which is to replace the pair of notions of relatedness and homogeneity that constitute it now, while retaining the overall effect.

Definition 2 (Entailment). *Let Γ be a finite possibly empty set of sentences.*

$\Gamma \models \varphi$ iff for all i : if $i \models \psi$, for all $\psi \in \Gamma$, then $i \models \varphi$.

As usual, equivalence of two sentences is characterized by mutual entailment, i.e., two sentences are equivalent iff they are supported by the same states.

It is a characteristic feature of the semantics that all sentences of the language have the property of persistence: if a sentence is supported by a state, it is also supported by all of its substates.

Theorem 1 (Persistence). *If $i \models \varphi$, then for all $j \subseteq i$: $j \models \varphi$.*

The semantics collapses into a classical semantics if we consider support of a sentence φ in a final state $\{v\}$. For $\{v\} \models \varphi$, all clauses in the semantics boil down to $v \models \varphi$, where \models is the classical satisfaction relation. For this reason, and for the sake of readability, we write $v \models \varphi$ instead of $\{v\} \models \varphi$.

5 Classical Sentences and Disjunction

Persistence implies that if a state supports a sentence, then so do all of its final substates, i.e., if $i \models \varphi$, then for all $v \in i$: $v \models \varphi$. If the reverse would also hold for all sentences of the language, then all sentences would be classical according to the following definition:

Definition 3 (Classical Sentences).

φ is classical iff if for all $v \in i$: $v \models \varphi$, then $i \models \varphi$

The semantics would collapse into classical semantics if all sentences were classical. But that is not so.

Fact 1 (Disjunction is not classical). $p \vee q$ is not classical.

Proof. Let i be a state $\{v, u\}$, where $v(p) = 1$ & $v(q) = 0$ and $u(p) = 0$ & $u(q) = 1$. It holds that $v \models p \vee q$, because $v \models p$, and $u \models p \vee q$, because $u \models q$, but $i \not\models p \vee q$, because $i \not\models p$, since $u \in i$ & $u(p) = 0$, and $i \not\models q$, since $v \in i$ & $v(q) = 0$. This shows that there is a state i such that for all $v \in i$: $v \models p \vee q$, whereas $i \not\models p \vee q$. Hence, $p \vee q$ is not classical.

If it were not for disjunction, the semantics would be classical. The disjunction-free fragment of the language contains only classical sentences.

Fact 2. *All sentences of the disjunction-free fragment of L are classical.*

Proof. The proof is by induction on the length of a formula of the disjunction free fragment of the language, and proceeds by showing that:

1. \perp is classical.
2. For all $p \in P$: p is classical.

3. If φ and ψ are classical, then $\varphi \wedge \psi$ is classical.
4. If ψ is classical, then $\varphi \rightarrow \psi$ is classical.

Note that a conditional sentence is classical as soon as its consequent is, irrespective of the nature of the antecedent. Since \perp is classical, this implies that $\neg\varphi$, defined as $\varphi \rightarrow \perp$, is classical, and hence $!\varphi$, defined as $\neg\neg\varphi$ is classical. One thing this means is that since we have seen that $p \vee q$ is not classical, and $!(p \vee q)$ is, the two cannot be equivalent. This is just a preview of some basic features of the semantics, more discussion on these matters will follow below.

We introduce a notation for classical propositions and classical updates.

Definition 4 (Classical propositions and updates).

1. $[\varphi] = \{v \in \omega \mid v \models \varphi\}$
2. $i[\varphi] = i \cap [\varphi]$

We can look upon $[\varphi]$ as the classical meaning of φ , the proposition that is classically expressed by φ , and we can look upon $i[\varphi]$ as the result of a classical update of a state i with a sentence φ . (Note: it might be that $i[\varphi] = \emptyset$, and hence that $i[\varphi]$ is not a state, under the present notion of what a state is.)

6 Two Semantic Properties, Four Semantic Categories

We define the semantic properties of informativeness and inquisitiveness of sentences relative to states, where we obtain corresponding absolute notions by requiring there to be some state where the sentence has the property.²

Definition 5 (Informativeness, and Inquisitiveness).

1. φ is informative in i iff for some $v, u \in i$: $v \models \varphi$ & $u \not\models \varphi$;
 φ is informative iff φ is informative in some state i .
2. φ is inquisitive in i iff for some $j \subseteq i$: $j \not\models \varphi$ & for all $v \in j$: $v \models \varphi$.
 φ is inquisitive iff φ is inquisitive in some state i .
3. φ is significant (in i) iff φ is informative (in i) or φ is inquisitive (in i).

It follows immediately from the definitions of inquisitiveness and persistence that inquisitiveness is a non-classical semantic feature.

Fact 3 (Inquisitive is not classical). φ is classical iff φ is not inquisitive.

² Thanks to Anna Szabolcsi, and the other participants in the class, for suggesting to exclude contradictions from being informative. I had things like this in a draft of the Tbilisi paper, but the reason for changing that in the final version, does no longer apply. The reason was that in the Tbilisi paper I wanted to define questions as non-informative sentences. But here, I also require that they are inquisitive. So, now, in the Tractarian way, contradictions and tautologies are again viewed as two sides of the same insignificant coin.

Since we saw that $p \vee q$ is not classical, whereas $!(p \vee q)$ is, the latter is not inquisitive, whereas the former is. That is why we called $!\varphi$ the non-inquisitive closure of φ . We will see that in terms of informativeness there is no difference between φ and $!\varphi$. This implies that $p \vee q$ is ‘more significant’ than $!(p \vee q)$, in the sense that there are states where $p \vee q$ is significant in being inquisitive, whereas $!(p \vee q)$ is not, since it is a state where the disjunction is not informative. (The state used in the proof of Fact 1 is such a state).

Both informativeness of φ in i and inquisitiveness of φ in i require that there is a substate of i that does not support φ , which by persistence means that $i \not\models \varphi$. So, significance, being informative or inquisitive, implies non-support, which indicates that being supported by the current state is a negative feature from a conversational point of view.

Both informativeness of φ in i and inquisitiveness of φ in i also require that there is a substate of i that does support φ , which by persistence means that there is at least some $v \in i$: $v \models \varphi$. So, significance implies consistency, which indicates that being inconsistent with the current state is also a negative feature from a conversational point of view.

Fact 4 (Significance). φ is significant in i iff $i \not\models \varphi$ and for some $v \in i$: $v \models \varphi$.

The absolute versions of the notions of informativeness and inquisitiveness are defined in terms of there being some state where a sentence has the property in question. Persistence guarantees that if a sentence has one of these properties in some state i , then it has it in any state j : $i \subseteq j$ as well, and hence in ω . Thus, an alternative way to characterize the absolute notion of significance is as follows:

Fact 5. φ is significant iff φ is significant in ω .

The only sentences that are never significant are contradictions and tautologies.

Definition 6 (Tautologies and Contradictions).

1. φ is a tautology iff for all i : $i \models \varphi$.
2. φ is a contradiction iff for no i : $i \models \varphi$.

It should be noted though, that whereas the notion of a contradiction is classical, the notion of a tautology is not. In inquisitive semantics, being a tautology does not just mean not being informative, but also not being inquisitive. A classical tautology, i.e., a non-informative sentence, can very well be inquisitive, and hence not be a tautological sentence, but a significant one. This should be kept in mind in evaluating the following fact:

Fact 6. φ is insignificant iff φ is a tautology or a contradiction.

In terms of the properties of informativeness and inquisitiveness, next to insignificant sentences, we distinguish three semantic categories of significant sentences:

Definition 7 (Assertions, Questions, and Hybrids). *Assertions, questions, hybrids, and insignificant sentences, are sentences which have or lack the properties of informativeness and inquisitiveness as indicated in the following table:*

φ	informative	inquisitive
insignificant	–	–
assertion	+	–
question	–	+
hybrid	+	+

Given that non-inquisitiveness and being a classical sentence coincide, a sentence is classical iff it is an insignificant sentence, a tautology or a contradiction, or an assertion. As we noted above, a classical tautology, a non-informative sentence, can very well be significant in being inquisitive. As can be read from the table, if a sentence is a question, it is a classical tautology. Plain disjunctions are hybrids:

Fact 7. $p \vee q$ is a hybrid sentence

Proof. Consider state i in the proof of Fact 1 and add an index w to it where $w(p) = w(q) = 0$. This is a state where $p \vee q$ is inquisitive and informative. Since there is such a state, $p \vee q$ is a hybrid.

7 Possibilities for Sentences

Given that all sentences are persistent, we can sensibly talk about a \subseteq -maximal substate of a state i that support a sentence φ , which we call a *possibility for φ in i* . It is characteristic for the semantics that there may be more than one possibility for a sentence.

Definition 8 (Possibilities for a Sentence in a State).

$$i[\varphi] = \{j \subseteq i \mid j \models \varphi \text{ and for all } k \subseteq i: \text{ if } k \models \varphi \ \& \ j \subseteq k, \text{ then } j = k\}.$$

We use an update-like notation, but unlike what is standardly the case, the output of $i[\varphi]$ is not a state, is not the substate of i that results from updating i with φ , but a (possibly empty) set of substates of i . From a pragmatic point of view, we look upon these possibilities as embodying a proposal to update the current state i to one of these possibilities.

Instead of $\omega[\varphi]$ we write $[\varphi]$, and we call $[\varphi]$ the *proposition expressed by φ* , and for $i \in [\varphi]$, we say that i is a *possibility for φ* . We use $\#i[\varphi]$ to denote the cardinality of the set of possibilities for φ in i .

The proposition expressed by a sentence fully characterizes its meaning, when the latter is viewed as the set of all states that support φ :

Fact 8 (Propositions and Meanings).

$i \models \varphi$ iff there is some j such that $j \in [\varphi]$ and $i \subseteq j$.

Since propositions characterize meaning, we are entitled to say that two sentences are equivalent if they express the same proposition.

Definition 9 (Equivalence). φ and ψ are equivalent iff $[\varphi] = [\psi]$.

And, as is to be expected since propositions characterize meaning, we can also formulate the entailment relation between sentences in terms of the propositions they express: φ entails ψ iff every possibility for φ is included in a possibility for ψ .

Fact 9 (Entailment).

$\varphi \models \psi$ iff for all $i \in [\varphi]$: there is some $j \in [\psi]$ such that $i \subseteq j$.

We give two examples to illustrate the definition of the possibilities for a sentence.

Example 1. Consider again the state $i = \{v, u\}$ we used in the proof of Fact 1, where $v(p) = 1$ & $v(q) = 0$ and $u(p) = 0$ & $u(q) = 1$. Small as i is, there are two possibilities for $p \vee q$ in i : $i[p \vee q] = \{\{v\}, \{u\}\}$. We could also write $i[p \vee q] = \{i[p], i[q]\}$, using the classical update notion introduced in Def. 4.

Example 2. The proposition expressed by $p \vee q$ also contains two possibilities: $[p \vee q] = \{[p], [q]\}$. Both the proposition $[p]$ classically expressed by p , and the proposition $[q]$ classically expressed by q are sets of indices, states that support $p \vee q$. Any other state that does, is a substate of one of these two. So, these are the maximal states that support $p \vee q$, and hence the possibilities for $p \vee q$.

8 Note on the Hierarchy of Alternativehood.

It can be noted that the two possibilities for $p \vee q$ overlap. Any index v such that $v(p) = v(q) = 1$ is included in both possibilities for $p \vee q$.

The definition of the possibilities for a sentence embodies a specific notion of what a set \mathcal{P} of *alternative* possibilities is: for any two possibilities $i, j \in \mathcal{P}$, there should be a part $i' \subseteq i$ and $j' \subseteq j$ such that $i' \not\subseteq j$ and $j' \not\subseteq i$. We call this notion *Ivano Ciardelli alternatives*, or more shortly *IC-* or *\mathcal{C} -alternatives*.

The notion of \mathcal{C} -alternativehood is a rather weak notion. The strongest notion is the one that requires that no two possibilities in \mathcal{P} overlap, i.e., that for any two possibilities $i, j \in \mathcal{P}$: $i \cap j = \emptyset$. We call such alternatives *Brumwell Alternatives*, or more shortly *B-alternatives*, and a little less shortly *block-alternatives*.

On the other side of the spectrum, the weakest conceivable notion is where we only require that for any two $i, j \in \mathcal{P}$: $i \neq j$. The short name for this would be *\neq -alternatives*. A bit longer, and I might be twisting things a bit, is *Hamblin-alternatives*.

Weaker than block-alternativehood, but stronger than \mathcal{C} -alternativehood, is to require that for any possibility $i \in \mathcal{P}$, there is a part $i' \subseteq i$ such that for every other possibility $j \in \mathcal{P}$: $i' \not\subseteq j$, i.e., every alternative has a part that it doesn't share with any other alternative. This holds for the two possibilities for $p \vee q$, and (still to be discussed) for the two possibilities for a conditional question like $p \rightarrow ?q$. These we call *Velissaratou-alternatives*, or shortly *V-alternatives*.

Another notion, in-between V-alternativehood and \mathcal{C} -alternativehood, is that there is an $i \in \mathcal{P}$ such that for no $j \in \mathcal{P}$: $i \subseteq j$, i.e., there is a proper part of

the whole space covered by \mathcal{P} , that is not part of any of the possibilities in \mathcal{P} . A disjunction of questions like $?p \vee ?q$ exemplifies this type of alternativehood.

That the nature of such questions has caused pain and pleasure in the development of inquisitive semantics and pragmatics is *not* the reason to call this *SM-alternativehood*, which is short for *Salvador Mascarenhas alternativehood*. The reason is that both Salvador and SM-alternativehood are intimately related to the so-called ‘pair-semantics’, and that he was the first to see clearly that the set of possibilities $\mathcal{P} = \{\{v, u\}, \{u, w\}, \{w, v\}\}$, which are \mathcal{C} -alternatives as generated by the present ‘general semantics’, are not SM-alternatives, and is not generated by the pair-semantics as it can be found in the Tbilisi paper, and in Salvador’s MSc thesis. (It ‘collapses’ into $\{\{v, u, w\}\}$.) The general semantics embodies a weaker notion of alternativehood than the pair-semantics did.

This Hierarchy of Alternativehood is theoretically and empirically of interest. Natural language seems to prefer stronger over weaker alternatives. The weaker set of alternatives a proposition \mathcal{P} embodies, the more difficult it becomes to directly express \mathcal{P} in natural language. And both in semantics and pragmatics it has been observed that interpretation tends to ‘strengthen alternatives’. For $p \vee q$ such a strengthening means to eliminate the overlap of its possibilities $[p]$ and $[q]$, which leads to the exclusive interpretation of the disjunction. Such a strengthening also has to be called upon to arrive at an interpretation for $p \vee q \vee (p \wedge q)$ which differs from the interpretation of $p \vee q$.

9 Informativeness and Inquisitiveness Revisited

That there is more than one possibility for a sentence (in a state) corresponds to inquisitiveness of the sentence (in that state). Informativeness of a sentence φ in a state i requires that the union of the possibilities for φ in i is neither empty, nor equals i . The union of the possibilities for a sentence φ in a state i equals the output of the classical update of i with φ , which we defined to be equal to the intersection of i with the proposition classically expressed by φ (Def. 4).

Fact 10. $\bigcup i[\varphi] = i[\varphi]$

For the proposition expressed by φ this means that the union of the possibilities for φ equals the proposition that φ classically expresses: $\bigcup[\varphi] = [\varphi]$. In other words, as far as informativeness is concerned, the semantics remains classical.

We call the complement of the union of the possibilities for a sentence φ , if it is not empty, *the possibility that φ excludes*, and if it is empty, we say that φ *excludes no possibility*. We introduce the following notation:

Definition 10 (Exclusion Set). $i[\varphi] = \emptyset$, if $i[\varphi] = i$, else $i[\varphi] = \{i - i[\varphi]\}$.

We defined $i[\varphi]$ as being a set of possibilities, even though it can contain at most one. This brings the exclusion set in line with the set of possibilities for φ , which makes it easy to state things like:

Fact 11. $\bigcup(i[\varphi] \cup i[\varphi]) = \omega$.

Note that, given that $i[\varphi]$ is, $i[\varphi] \cup i[\varphi]$ cannot fail to be a set of \mathcal{C} -alternatives. This is obviously so in case $\#(i[\varphi]) = 0$, and if $\#(i[\varphi]) = 1$, the possibility it contains is a block-alternative, and hence a \mathcal{C} -alternative, of any possibility in $i[\varphi]$.

Having this little machinery in place, we can characterize informativeness, inquisitiveness, and significance, in terms of the cardinality of the possibilities for and excluded by a sentence.

Fact 12 (Informativeness and Inquisitiveness).

1. φ is informative in i iff $\#i[\varphi] \geq 1$ and $\#i[\varphi] = 1$.
2. φ is inquisitive in i iff $\#i[\varphi] \geq 2$.
3. φ is significant in i iff $\#(i[\varphi] \cup i[\varphi]) \geq 2$

Note that $\#(i[\varphi] \cup i[\varphi])$ is the same as $\#i[\varphi] + \#i[\varphi]$. We can also characterize the semantic categories of sentences in terms of the cardinality of the possibilities for and excluded by a sentence, this time distinguishing between the two insignificant categories of contradictions and tautologies.

φ	$\#[\varphi]$	$\#[\varphi]$	$\#([\varphi] \cup [\varphi])$
contradiction	0	1	1
tautology	1	0	1
assertion	1	1	2
question	≥ 2	0	≥ 2
hybrid	≥ 2	1	≥ 3

10 Classical Sentences

Given that (i) $\bigcup [\varphi] = [\varphi]$ (Fact 10), and that (ii) $\bigcup([\varphi] \cup [\varphi]) = \omega$ (Fact 11), and (iii) the cardinality facts given in the table above for classical sentences, i.e., contradictions, tautologies, and assertions, we can conclude the following:

Fact 13 (Classical Propositions).

1. φ is a contradiction iff $[\varphi] = \emptyset$ and $[\varphi] = \{\omega\}$.
2. φ is a tautology iff $[\varphi] = \{\omega\}$ and $[\varphi] = \emptyset$.
3. φ is an assertion iff $[\varphi] = \{[\varphi]\}$ and $[\varphi] = \{[\neg\varphi]\}$.

In the characterization of assertions $[\neg\varphi]$ occurs as the possibility that an assertion excludes. It doesn't really matter that we have not said much yet about the interpretation of negation in inquisitive semantics (it is the next topic), since $[\neg\varphi]$ is the *classical* interpretation of negation. In terms of that, we know quite generally, for any sentence φ , that *if* φ excludes a possibility, it must be the possibility that corresponds with the proposition classically expressed by its negation $\neg\varphi$.

Note on the absence of the absurd state. We have taken states to be non-empty sets of indices. There is no absurd state \emptyset around. In the semantics not much changes if we do allow for the absurd state. (Only the clause for \perp would say that it is only supported by the absurd state, whereas now it says that no state supports \perp .)

But where it does make a difference, is in the outcome of what the possibilities for a sentence are. We would get that $\lfloor \perp \rfloor = \{\emptyset\}$, instead of $\lfloor \perp \rfloor = \emptyset$, and, more generally, for any informative sentence, we would get that $\lfloor \varphi \rfloor = \{\emptyset\}$ where we now get $\lfloor \varphi \rfloor = \emptyset$. At certain points things get a bit more complicated by not allowing for the absurd state. We can't say, for example:

X If φ is classical, then $\lfloor \varphi \rfloor = \{\lfloor \varphi \rfloor\}$ and $\lceil \varphi \rceil = \{\lceil \neg \varphi \rceil\}$.

This works alright for assertions φ , because since these are informative, we know that $\lfloor \varphi \rfloor \neq \emptyset$ and $\lceil \neg \varphi \rceil \neq \emptyset$. But this is not so for contradictions and tautologies. So, under the formulation *X* of what the interpretation of classical sentences in general would be, we would get that $\lfloor \perp \rfloor = \{\emptyset\}$ instead of $\lfloor \perp \rfloor = \emptyset$, and similarly for $\lceil \top \rceil$. And $\{\emptyset\}$ is not a set of states when the absurd state is not allowed. (\emptyset is, of course, a set of states, the empty set of states.) If we do allow for the absurd state, *X* is fine. So, here we complicate things. On the other hand, Fact 13 formulates quite nicely how things are for classical sentences, we only have to separate out the three cases.

At other points, however, things get less elegant if we do allow for the absurd state. The one I find decisive is that it would no longer generally hold that $i\lfloor \varphi \rfloor \cup i\lceil \varphi \rceil$ is a set of \mathcal{C} -alternatives. (And as you may already have guessed: $i\lceil ?\varphi \rceil = i\lfloor \varphi \rfloor \cup i\lceil \varphi \rceil$.) In case one of the two is \emptyset , we would end up with a set which contains \emptyset together with one or more non-empty sets of which \emptyset is a proper subset. So, $i\lfloor \varphi \rfloor \cup i\lceil \varphi \rceil$ would not be a set of \mathcal{C} -alternatives. Exit the absurd state.

11 Conditionals and Entailment

Using the notion of the possibilities for a sentence we can reformulate the interpretation of conditional sentences as: a state i supports a conditional sentence $\varphi \rightarrow \psi$ iff every possibility for φ in i is included in some possibility for ψ in i .

Fact 14 (Conditionals).

$i \models \varphi \rightarrow \psi$ iff for all $j \in i\lfloor \varphi \rfloor$ there is some $k \in i\lceil \psi \rceil$: $j \subseteq k$

Proof. The original clause in the semantics quantifies over all substates of i that support the antecedent φ . By persistence it suffices to consider the maximal substates of i that support φ , which are the possibilities for φ in i . That such possibilities j for φ are to support ψ can be formulated as: j is included in some possibility k for ψ (Fact 8). Since j will be included in i , it makes no difference if we also restrict k to the possibilities for ψ in i .

From this reformulation of the interpretation of conditionals and the formulation of entailment in terms of propositions (Fact 9), it follows straightforwardly that:

Theorem 2. $\models \varphi \rightarrow \psi$ iff $\varphi \models \psi$

This is, of course, a rather standard logical feature. But we are not dealing with a very standard logical language. The single syntactic structure of conditional sentences covers assertions, questions and hybrids, as antecedent and consequent. But, at least, what the antecedent is like makes no difference in case the consequent is classical. As we claimed earlier (Fact 2.4), but didn't show:

Fact 15. *If ψ is classical, then $\varphi \rightarrow \psi$ is classical.*

Proof. (In which we shall also meet the interpretation of $\neg\varphi$, defined as $\varphi \rightarrow \perp$.) If the consequent ψ of a conditional $\varphi \rightarrow \psi$ is classical, i.e., if ψ is not inquisitive, then there is at most a single possibility for ψ , in any state i .

We first consider the case where there is no possibility for ψ in i . (This holds for \perp in any i .) Then it can only be the case that every possibility for φ in i is included in some possibility for ψ in i , if there is no possibility for φ in i . And, by persistence, that can only be the case if for no $v \in i$: $v \models \varphi$. In other words, if $i[\psi] = \emptyset$ (for all i : $i[\perp] = \emptyset$), then $i \models \varphi \rightarrow \psi$ iff for all $v \in i$: $i \models \varphi \rightarrow \psi$ iff for no $v \in i$: $v \models \varphi$. (So, $i \models \varphi \rightarrow \perp$, i.e., $i \models \neg\varphi$ iff for no $v \in i$: $v \models \varphi$.)

Next we consider the remaining case where there is one possibility k for ψ in i , where it must be the case that $k = \{v \in i \mid v \models \psi\}$. If we now have to consider whether every possibility $j \in i[\varphi]$ is included in k , we can just as well take their union, which is the set of indices $\{v \in i \mid v \models \varphi\}$. So, to inspect whether $i \models \varphi \rightarrow \psi$, we just end up checking for every $v \in i$ whether if $v \models \varphi$, then $v \models \psi$, i.e., whether for all $v \in i$: $v \models \varphi \rightarrow \psi$.

So, under the assumption that ψ is classical, both in case there is and in case there is no possibility for ψ in i , to see whether $i \models \varphi \rightarrow \psi$, it suffices to inspect whether for all $v \in i$: if $v \models \varphi$, then $v \models \psi$. By the definition of what a classical sentence is (Def 3), this means that given that ψ is classical $\varphi \rightarrow \psi$ is classical.

In combination with Theorem 2, the fact that a conditional sentence is classical as soon as its consequent is, also gives us important information about the entailment relation, and about the equivalence relation, which corresponds to mutual entailment. And for classical sentences, the equivalence relation behaves classically.

Fact 16 (Classical Entailment and Equivalence).

1. *If ψ is classical, then $\Gamma \models \psi$ iff $\Gamma \models_{class} \psi$.*
2. *If φ and ψ are classical, then $\varphi \equiv \psi$ iff $\varphi \equiv_{class} \psi$*

The entailment relation characterizes classical entailment as soon as the conclusion is a classical sentence, irrespective of the semantic nature of the premisses.

12 Negation and Significance

As we have seen as a side effect in the proof that a conditional sentence is classical as soon as its consequent is, negation behaves classically.

Fact 17 (Negation). $i \models \neg\varphi$ iff for no $v \in i$: $v \models \varphi$.

Although negation is classical, or better *because* negation is classical, $\neg\varphi$ has the special effect (inherited from its conditional background), of turning non-classical inquisitive sentences into classical non-inquisitive sentences. Negation standardly turns contradictions into tautologies, and *vice versa*, and turns assertions into assertions that assert the opposite:

Fact 18. If φ is classical, then $i[\neg\varphi] = i[\varphi]$ and $i[\neg\varphi] = i[\varphi]$.

Negation only concerns information. What negation does to an inquisitive sentence φ depends on whether φ is also informative or not, i.e., depends on whether φ is a hybrid or a question. Questions being not informative, like tautologies, turn under negation into contradictions. (See below.) Hybrids being informative, like assertions, turn under negation into assertions.

Fact 19. If φ is a hybrid, then $i[\neg\varphi] = i[\varphi]$ and $i[\neg\varphi] = \{i[\varphi]\} = \{\bigcup i[\varphi]\}$.

The single possibility for the negation of a hybrid sentence, like $\neg(p \vee q)$, equals the possibility that the hybrid sentence excludes. The possibility that the negation of a hybrid sentence excludes consists of the union of the possibilities for the hybrid sentence, i.e., equals the proposition that the hybrid classically expresses:

$$i[\neg(p \vee q)] = \{i[p] \cup i[q]\} = \{i[p \vee q]\} = \{\{v \in i \mid v(p) = 1 \text{ or } v(q) = 1\}\}.$$

As in classical logic, we can characterize significance as follows:

Fact 20 (Significance). φ is significant iff $\not\models \varphi$ and $\not\models \neg\varphi$

Since $\neg\varphi$ is classical, $\not\models \neg\varphi$ behaves classically, and means that φ is not a contradiction. Likewise $\not\models \varphi$ means that φ is not a tautology, but that is not a classical notion. It does not mean that there must be a state where φ is informative in that φ excludes a possibility in that state, it is also sufficient if φ is inquisitive in some state.

The difference between inquisitive significance and classical significance is pretty much summarized by the following fact about negation and significance:

Fact 21 (Negation and Significance).

If $\neg\varphi$ is significant, then φ is significant.

The point being that classically this also runs in the other direction, classically φ is significant if and only if $\neg\varphi$ is significant.

13 Non-Inquisitive Closure

Since $!\varphi$ is defined as $\neg\neg\varphi$, the non-inquisitive closure $!\varphi$ of φ is classical.

Fact 22 (Non-inquisitive closure). $i \models !\varphi$ iff for all $v \in i$: $v \models \varphi$

It follows immediately from Fact 17 that the non-inquisitive closure of a classical sentence has no semantic effect. The non-inquisitive closure of a question is a tautology, and hybrids are turned into assertions.

$$i[!(p \vee q)] = i[\neg(p \vee q)] = \{i[p] \cup i[q]\} = \{i[p \vee q]\}.$$

The general pattern of non-inquisitive closure is:

Fact 23 (Pattern of non-inquisitive closure).

1. $i[!\varphi] = \emptyset$ if $i[\varphi] = \emptyset$, else $i[\varphi] = \{\bigcup i[\varphi]\} = \{i[\varphi]\}$
2. $i[!\varphi] = i[\varphi]$

What $!\varphi$ excludes is the same as what φ excludes. If there is no possibility for φ , then neither is there one for $!\varphi$. And if there are possibilities for φ , there is a single possibility for $!\varphi$ which is the union of the possibilities for φ , i.e., it is the proposition that is classically expressed by φ . Summing up:

Fact 24 (Properties of non-inquisitive closure).

1. $!\varphi$ is classical.
2. $!\varphi$ is equivalent with φ iff φ is classical.
3. $!\varphi$ is equivalent with $!\varphi$.
4. $!\varphi$ is an assertion iff φ is informative.
5. If $!\varphi$ is significant, then φ is significant.

The last item in this list is the most important one. (Well, what is important is that the reverse does not hold.) The crucial feature is that if $\not\models !\varphi$, then $\not\models \varphi$. The other aspect of significance, that if $\not\models \neg!\varphi$, then $\not\models \neg\varphi$, is obviously the case. Since both $\neg!\varphi$ and $\neg\varphi$ are assertions and classically equivalent, they are equivalent in inquisitive semantics as well. That if $\not\models !\varphi$, then $\not\models \varphi$, means that if $\models \varphi$, then $\models !\varphi$, which follows immediately from the fact below, which we will use to prove that inquisitive validity implies classical validity. The essential entailment fact about non-inquisitive closure is:

Lemma 1 (Non-Inquisitive Closure). $\varphi \models !\varphi$

Proof. Since $!\varphi$ is $\neg\neg\varphi$, it holds classically that $\varphi \equiv_{\text{class}} !\varphi$. Hence it holds that $\varphi \models_{\text{class}} !\varphi$. Given that $!\varphi$ is classical, it then holds also that $\varphi \models !\varphi$. since we have already seen that entailment boils down to classical entailment, when the conclusion is an assertion. (The reverse only holds if φ is a classical sentence.)

Theorem 3. If $\models \varphi$, then $\models_{\text{class}} \varphi$

Proof. Suppose that $\models \varphi$. We have just seen that $\varphi \models !\varphi$. So, given that $\models \varphi$, it also holds that $\models !\varphi$. Since $!\varphi$ is an assertion $\models !\varphi$ implies that $\models_{\text{class}} !\varphi$. Classically $\varphi \equiv_{\text{class}} !\varphi$. Hence, given that $\models_{\text{class}} !\varphi$, it also holds that $\models_{\text{class}} \varphi$. This means that we have shown that If $\models \varphi$, then it also holds that $\models_{\text{class}} \varphi$.

It follows straightforwardly from Theorem 3 and Theorem 2 that If $\Gamma \models \varphi$, then $\Gamma \models_{\text{class}} \varphi$. So, inquisitive entailment guarantees classical entailment. A typical example of a classical entailment that has no inquisitive counterpart concerns the non-inquisitive closure of disjunction.

Fact 25. $!(p \vee q) \not\models p \vee q$.

Proof. The single possibility $[p] \cup [q]$ for $!(p \vee q)$ is not included in either of the two possibilities $[p]$ and $[q]$ for $p \vee q$.

Of course, it does hold that $\neg\neg(p \vee q) \models_{\text{class}} p \vee q$, and hence that $!(p \vee q) \models_{\text{class}} p \vee q$.

14 Questions

Given that $?\varphi$ is defined as $\varphi \vee \neg\varphi$, and given the interpretation of negation, we get that the non-informative closure $?\varphi$ of a sentence φ is interpreted as follows:

Fact 26 (Non-informative closure). $i \models ?\varphi$ iff $i \models \varphi$ or for no $v \in i$: $i \models \varphi$.

The semantic effect of non-informative closure clearly appears in:

Fact 27. $i[?\varphi] = i[\varphi] \cup i[\neg\varphi]$; and $i[?\varphi] = \emptyset$

A sentence $?\varphi$ never excludes a possibility, and the possibilities for $?\varphi$ are obtained by adding the possibility excluded by φ , if there is one, to the possibilities for φ , if such there are. There is always at least one possibility for $?\varphi$.

If there is a single possibility for $?\varphi$ in a state i , then $?\varphi$ is not inquisitive in i . That can be the case because there is already only one possibility in the proposition expressed by $?\varphi$. In that case we have that $[?\varphi] = \{\omega\}$, which is so iff φ is not significant, i.e., both in case φ is a tautology, and in case φ is a contradiction.

As soon as φ is significant $?\varphi$ expresses an inquisitive proposition, i.e., φ is a question. Of course, that $?\varphi$ as such is a question, does not yet guarantee that φ is inquisitive in a state i . It might be that the issue $?\varphi$ raises is already resolved in i . (Like assertions need not necessarily be informative in a state i , and a hybrid can be neither informative nor inquisitive in a state i .)

The most interesting property of $?\varphi$ is the third one in the list below, which says in so many words that the defining feature of the operator $?$ is that $?\varphi$ cannot be negated without becoming a meaningless contradiction.

Fact 28 (Properties of non-informative closure).

1. $?\varphi$ is not informative.
2. $\neg?\varphi$ is a contradiction.
3. $?\varphi$ is equivalent with φ iff $\neg\varphi$ is a contradiction.
4. $??\varphi$ is equivalent with $?\varphi$
5. $?\varphi$ is a question iff φ is significant.

The following list exemplifies: (1) atomic polar question; (2) hybrid disjunction; (3) alternative question; (4) polar disjunctive question; (5) conjunction of questions; (6) disjunction of questions; (7) hybrid conjunction of assertion and question; (8) disjunction of assertion and question.

Fact 29 (Examples).

1. $[?p] = \{[p], [\neg p]\}$
2. $[p \vee q] = \{[p], [q]\}$; and $[p \vee q] = \{[\neg p \wedge \neg q]\}$
3. $[?(p \vee q)] = \{[p], [q], [\neg p \wedge \neg q]\}$
4. $[?!(p \vee q)] = \{[(p \vee q), [\neg p \wedge \neg q]]\}$
5. $[?(p \wedge ?q)] = \{[p \wedge q], [\neg p \wedge q], [p \wedge \neg q], [\neg p \wedge \neg q]\}$
6. $[?(p \vee ?q)] = \{[p], [\neg p], [q], [\neg q]\}$
7. $[(p \wedge ?q)] = \{[p \wedge q], [p \wedge \neg q]\}$; and $[(p \wedge ?q)] = \{[\neg p]\}$
8. $[(p \vee ?q)] = \{[p], [q], [\neg q]\}$

Under the classical notion of entailment all questions entail each other, and hence all questions are classically equivalent, and equivalent with \top . This also means, of course, that any assertion or hybrid classically properly entails any question.

Under the inquisitive entailment notion things are different, among questions entailment measures inquisitiveness.

Fact 30 (Inquisitive entailment).

$? \varphi \models ? \psi$ iff for all i : if $? \varphi$ is inquisitive in i , then $? \psi$ is inquisitive in i .

Concerning the six questions in the list of examples above, the conjunction of questions (5) entails the atomic polar question (1), which in turn entails the disjunction of questions (6). The conjunction of questions (5) also entails the alternative question (3), which in turn entails the polar disjunctive question (4), and the disjunction of questions (6). Finally, The conjunction of questions (5) also entails the disjunction of an assertion and question (8), which also entails the disjunction of questions (6).

No question can entail an assertion or hybrid. The hybrid disjunction (2) entails the alternative question (3). The hybrid conjunction of a question and assertion (7) all other sentences in the list.

If an assertion $! \varphi$ entails a question $? \psi$, meaning that the single possibility for $! \varphi$ is included in one of the possibilities for $? \psi$, then $! \varphi$ fully resolves the issue $? \psi$ poses. However, this does not really correspond to a relevant notion of linguistic complete answerhood, unless you require that $! \varphi \models ? \psi$ and for no non-equivalent $! \chi$ such that $! \varphi \models \varphi$: $\chi \models ? \psi$.

Also, entailment does not cover partial answerhood. What works for question for which the possibilities are block-alternatives, is to say that $! \varphi$ is a partial answer to $? \psi$, in case $? \psi \models ? ! \varphi$, i.e., of te question entails the yes/no-question that correspond to the answer. But that doesn't work as a notion of answerhood for questions which embody weaker notions of alternativehood. For example, p is an answer to $?(p \vee q)$, but $?(p \vee q) \not\models ? p$.

What entailment also does not appropriately cover, for weaker notions of alternativehood than block-alternatives, is the relation of subquestionhood. Although it does always hold that when $?\varphi \models ?\psi$ it means that $?\psi$ is easier to answer than $?\varphi$, what does not hold in general is that when $? \varphi \models ? \psi$ ψ is a related subquestion of $?\varphi$. A case in point is that whereas $?p \models ?p \vee q$, it intuitively does not hold that $?p \vee q$ is a subquestion of $?p$. The latter is an (atomic) polar question which typically do not have any subquestions.

15 Non-Classical Conditionals

Consider again the formulation of the interpretation of conditionals using the notion of the possibilities for a sentence.

$$i \models \varphi \rightarrow \psi \text{ iff for all } j \in i[\varphi] \text{ there is some } k \in i[\psi]: j \subseteq k$$

We have seen already above that if the consequent of a conditional is not inquisitive, then neither is the conditional as such. The first item in the fact below is another way to put this. Likewise, the second item tells us, that it is also the case that as soon as the consequent of a conditional is not informative, the conditional as such is not informative either. This is not so surprising, it corroborates the fact that as far as informativeness is concerned inquisitive semantics is conservative.

Fact 31 (Non-inquisitive and non-informative conditionals).

1. $\varphi \rightarrow !\psi$ is not inquisitive.
2. $\varphi \rightarrow ?\psi$ is not informative.

Proof. (Of second item.) For $\varphi \rightarrow ?\psi$ to be informative, there should be some v such that $v \not\models \varphi \rightarrow \psi$. That can only be the case if there is some $v: v \models \varphi$ and $v \not\models \psi$. But that contradicts that $?\psi$ is not informative.

Things get more interesting if we consider sentences where the consequent is not just guaranteed to be non-informative, but is also inquisitive, i.e., where the consequent of a conditional is a question. the simplest example is a conditional sentence like $p \rightarrow ?q$, i.e., $p \rightarrow (q \vee \neg q)$.

Fact 32 (Conditional Question). $p \rightarrow ?q$ is a question.

Proof. Let i be a state with indices $v, u \in i: v(p) = u(p) = 1$ & $v(q) = 1$ & $u(q) = 0$. Then there are two possibilities for $?q$ in i , $i[q]$ and $i[\neg q]$, where $v \in i[q]$ and $u \in i[\neg q]$. And there is one possibility for p in i , the possibility $i[p]$, where both $v, u \in i[p]$. Such a state i does not support $p \rightarrow ?q$. The interpretation of $p \rightarrow ?q$ requires that $i[p]$, and hence both v and u , are included in some possibility for $?q$ in i . But v and u are not both in one and the same possibility for $?q$. Of course both $v \models p \rightarrow ?q$ and $u \models p \rightarrow ?q$, but $\{v, u\} \not\models p \rightarrow ?q$. Hence $p \rightarrow ?q$ is inquisitive in i . So, $p \rightarrow ?q$ is an inquisitive sentence, and a non-informative sentence. Which shows that it is a question.

There are two possibilities for $p \rightarrow ?q$, which correspond to $p \rightarrow q$ and $p \rightarrow \neg q$. This also means that $p \rightarrow ?q$ is equivalent with $(p \rightarrow q) \vee (p \rightarrow \neg q)$.

Fact 33. $[p \rightarrow ?q] = \{[p \rightarrow q], [p \rightarrow \neg q]\} = [(p \rightarrow q) \vee (p \rightarrow \neg q)]$

The conditional $p \rightarrow (q \vee r)$, with a hybrid disjunction as consequent, is itself hybrid as well:

Fact 34. $[p \rightarrow (q \vee r)] = \{[p \rightarrow q], [p \rightarrow r]\}$, and $[p \rightarrow (q \vee r)] = \{[p \wedge \neg q \wedge \neg r]\}$.

Another interesting fact is that any conditional sentence is equivalent with the conjunction of the corresponding conditional question and conditional assertion.

Fact 35. $\varphi \rightarrow \psi$ is equivalent with $(\varphi \rightarrow ?\psi) \wedge (\varphi \rightarrow !\psi)$

For a longer story, see the Tbilisi paper.

16 Counting Possibilities

We have seen in previous sections where we discussed conditionals, that if the consequent is not an inquisitive sentence, then neither is the conditional as such, irrespective of the nature of the antecedent. This means, for example, that although the antecedent in $(p \vee q) \rightarrow r$ is a hybrid and hence inquisitive disjunction, since the consequent is an assertion, so is the sentence as a whole. It just receives its standard interpretation.

We have also seen that a conditional like $p \rightarrow (q \vee r)$, where this time the consequent is an inquisitive sentence, is itself inquisitive. It is a hybrid sentence, for which there are two possibilities, and a possibility which it excludes. The natural question to ask is: Is the fact that an inquisitive antecedent has no inquisitive effect in conditional sentence where the consequent is not inquisitive, inherited by conditionals with inquisitive consequents, or not?

If you inspect the interpretation of conditional sentences formulated in terms of possibilities for sentences, you will see that as soon as the consequent is inquisitive, inquisitiveness of the antecedent can have a significant effect. If there are several possibilities for the consequent, then, in principle, each of the possibilities for the antecedent can possibly be included in each of the possibilities for the consequent. That means that in total, with a possibilities for the antecedent, and c possibilities for the consequent, there are c^a possibilities for the sentence as a whole.

Once we start counting numbers of possibilities for conditionals, we may just as well do that for the other connectives. We denote by $\text{MAX}[\varphi]$, the maximum number of possibilities for φ . The counting goes as follows:

Fact 36 (Counting Possibilities).

1. $\text{MAX}[p] = 1$
2. $\text{MAX}[\neg\varphi] = 1$
3. $\text{MAX}[\varphi \vee \psi] = \text{MAX}[\varphi] + \text{MAX}[\psi]$

4. $\text{MAX}[\varphi \wedge \psi] = \text{MAX}[\varphi] \times \text{MAX}[\psi]$
5. $\text{MAX}[\varphi \rightarrow \psi] = \text{MAX}[\psi]^{\text{MAX}[\varphi]}$

And we note the following fact:

Fact 37. $\text{MAX}[(p \vee q \vee r) \rightarrow ?s] = 8$ and $\text{MAX}[!(p \vee q \vee r) \rightarrow ?s] = 2$

$$[(p \vee q \vee r) \rightarrow ?s] = \left\{ \begin{array}{l} [(p \vee q \vee r) \rightarrow s], \\ [(p \vee q \vee r) \rightarrow \neg s], \\ [((p \vee q) \rightarrow s) \wedge (r \rightarrow \neg s)], \\ [((p \vee q) \rightarrow \neg s) \wedge (r \rightarrow s)], \\ [((p \vee r) \rightarrow s) \wedge (q \rightarrow \neg s)], \\ [((p \vee r) \rightarrow \neg s) \wedge (q \rightarrow s)], \\ [((q \vee r) \rightarrow s) \wedge (p \rightarrow \neg s)], \\ [((q \vee r) \rightarrow \neg s) \wedge (p \rightarrow s)] \end{array} \right\}$$

$$[!(p \vee q \vee r) \rightarrow ?s] = \{ [(p \vee q \vee r) \rightarrow s], [(p \vee q \vee r) \rightarrow \neg s] \}$$

17 Preview

The plan for the next section(s) to come, after wrapping up the semantics part properly, is to introduce the logical notion of compliance (cf. the past part of the handout of February 25, or the last part of the Tbilisi paper for the shortest introduction), and discuss its relationship with the notion of entailment.