Dynamic Epistemic Logic of Questions

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Introduction & Motivation

Epistemic-Issue Models

Static Logic of Questions

Issue-Management Actions

Dynamic Logic of Questions

Extensions:
- Multi-Agent Scenarios
- Protocols

Further Research Topics

Questions are important because:

- They are ubiquitous in natural language and communication
- They are indispensable for understanding inquiry and discovery
- They play an essential part in human rational interaction
- They feature in many epistemic puzzles that founded DEL

Our approach will use standard DEL methodology and expand its research agenda by considering issue management actions

Previous approaches to questions:

- (Groenendijk & Stokhof 1997), (Groenendijk 2008)
- (Hintikka, Halonen & Mutanen 2001), (Hintikka 2007)
- (Baltag 2001), (Baltag & Smets 2009)
- (Unger & Giorgolo 2007), (van Eijck & Unger 2009)

Definition (Epistemic Issue Model)

A structure $M = \langle W, \sim, \approx, V \rangle$ *with:*
- W *is a set of possible worlds or states (epistemic alternatives),*
- $\sim$ *is an equivalence relation on* W *(epistemic indistinguishability),*
- $\approx$ *is an equivalence relation on* W *(the abstract issue relation),*
- $V : \mathtt{P} \to \wp(W)$ *is a valuation function mapping atoms to worlds.*

Definition (Static Language)

The language $\mathcal{L}_{\mathbf{EL_Q}}(\mathtt{P}, \mathtt{N})$ *is given by this inductive syntax rule:*

$$i \mid p \mid \bot \mid \neg\varphi \mid (\varphi \wedge \psi) \mid U\varphi \mid K\varphi \mid Q\varphi \mid R\varphi$$

$$i \mid p \mid \bot \mid \neg\varphi \mid (\varphi \wedge \psi) \mid U\varphi \mid K\varphi \mid Q\varphi \mid R\varphi$$

Definition (Interpretation)

Formulas are interpreted in models M at worlds w with the standard boolean and modal clauses and:

$$\begin{array}{lll} M \models_w K\varphi & \textit{iff} & \textit{for all } v \in W : w \sim v \textit{ implies } M \models_v \varphi, \\ M \models_w Q\varphi & \textit{iff} & \textit{for all } v \in W : w \approx v \textit{ implies } M \models_v \varphi, \\ M \models_w R\varphi & \textit{iff} & \textit{for all } v \in W : w \,(\sim \cap \approx)\, v \textit{ implies } M \models_v \varphi. \end{array}$$

$K\varphi$ describes the semantic information of an agent: "φ is known", "φ holds in all epistemically indistinguishable worlds"

$Q\varphi$ describes the current structure of the issue-relation: "φ holds in all issue-equivalent worlds"

$R\varphi$ is the 'resolving' modality describing what the agent would come to know after all the questions have been answered. It says: "φ holds in all worlds which are **both** epistemically indistinguishable and issue equivalent"

This static language can express useful notions:

- $U(Q\varphi \vee Q\neg\varphi)$ fact φ is settled by the structure of the current issue relation.
- $\widehat{K}(\varphi \wedge \widehat{Q}\neg\varphi)$ the agent considers it possible that fact φ is not settled by the current structure of the issue relation,
- $KQ\varphi \wedge \neg U(Q\varphi \vee Q\neg\varphi)$ locally, the agent knows that fact φ is settled but globally it is not,
- $\neg\widehat{U}(K\varphi \vee Q\varphi) \wedge UR\varphi$ fact φ is neither known nor settled by the issue-relation structure but it can become settled after a resolution action.

$$\mathbf{EL_Q} = \{\varphi \in \mathcal{L}_{\mathbf{EL_Q}} : \ \models \varphi\}$$

Axiomatic proof system for $\mathbf{EL_Q}$:

Customary epistemic-S5 axioms for knowledge:

1. $Kp \to p$ (Truth), $Kp \to KKp$, $\neg Kp \to K\neg Kp$ (Introsp$\pm$);

S5 axioms for the other two equivalence relations:

2. $p \to Q\widehat{Q}p$ (Symm), $p \to \widehat{Q}p$ (Rflx), $\widehat{Q}\widehat{Q}p \to \widehat{Q}p$ (Trns)
3. $p \to R\widehat{R}p$ (Symm), $p \to \widehat{R}p$ (Rflx), $\widehat{R}\widehat{R}p \to \widehat{R}p$ (Trns)

Customary axiom for the intersection modality:

4. $\widehat{K}i \wedge \widehat{Q}i \leftrightarrow \widehat{R}i$ (Intersection)

Standard system of modal (hybrid) logic with universal modality.

Standard system of hybrid logic with universal modality:

5. $\Box(p \to q) \to (\Box p \to \Box q), \Box \in \{UKRQ\}$ (Distribution)
6. $\neg\Box\neg p \leftrightarrow \Diamond p, \Diamond, \Box \in \{UKRQ\}$ (Duality)
7. $p \to U\widehat{U}p$ (Symm), $p \to \widehat{U}p$ (Rflx), $\widehat{U}\widehat{U}p \to \widehat{U}p$ (Trns),
8. $\widehat{U}i, \Diamond p \to \widehat{U}p, \Diamond \in \{KRQ\}$ (Inclusion)
9. $\Diamond(i \wedge p) \to \Box(i \to p), \Box \in \{UKRQ\}$ (Nominals)
10. From $\vdash_{\mathsf{PC}} \varphi$ infer φ (Prop), From φ and $\varphi \to \psi$ infer ψ (M P)
11. From φ infer $\Box\varphi$, for $\Box \in \{UKRQ\}$ (Necessitation)
12. From φ and $\sigma_{\mathrm{sort}}(\varphi) = \psi$ infer ψ, where σ_{sort} is sorted (sSbs)
13. From $i \to \varphi$ infer φ, for i not occuring in φ (Nam)
14. From $\widehat{U}(i \wedge \Diamond j) \to \widehat{U}(j \wedge \varphi)$ infer $\widehat{U}(i \wedge \Box\varphi)$, for $\Diamond \in \{KRQ\}$, $i \neq j$, and j not occuring in φ, (B G)

Basic principles are derivable in this system, for example:

$$U(Qp \vee Q\neg p) \quad \vdash_s \quad UU(Qp \vee Q\neg p) \quad \vdash_s \quad KU(Qp \vee Q\neg p)$$

(Introspection about the current public issue)

Theorem (Completeness of $\mathbf{EL_Q}$)

For every formula $\varphi \in \mathcal{L}_{\mathbf{EL_Q}}(\mathtt{P}, \mathtt{N})$ it is the case that:

$$\models \varphi \quad \textit{if and only if} \quad \vdash \varphi$$

Proof.

By standard techniques for multi-modal hybrid logic. $\square$

Dynamics of Information and Issues

Definition (Questions & Announcements)

An execution of a $\varphi?$ *action in model* M *results in a new model* $M_{\varphi?} = \langle W_{\varphi?}, \sim_{\varphi?}, \approx_{\varphi?}, V_{\varphi?}\rangle$. *Likewise, a* $\varphi!$ *action results in a changed model* $M_{\varphi!} = \langle W_{\varphi!}, \sim_{\varphi!}, \approx_{\varphi!}, V_{\varphi!}\rangle$, *with:*

$$\begin{aligned} W_{\varphi?} &= W \\ \sim_{\varphi?} &= \sim \\ \approx_{\varphi?} &= \approx \cap \stackrel{\varphi}{\equiv}_M \\ V_{\varphi?} &= V \end{aligned} \qquad\qquad \begin{aligned} W_{\varphi!} &= W \\ \sim_{\varphi!} &= \sim \cap \stackrel{\varphi}{\equiv}_M \\ \approx_{\varphi!} &= \approx \\ V_{\varphi!} &= V \end{aligned}$$

$$\text{where:} \quad \stackrel{\varphi}{\equiv}_M = \{(w,v) \mid \|\varphi\|^M_w = \|\varphi\|^M_v\}$$

The symmetry is not always complete:
$p!$ is executable only in worlds where it is *truthful*;
$p?$ is executable in every world, even those not satisfying p.

Figure: Effects of Asking Yes/No Questions

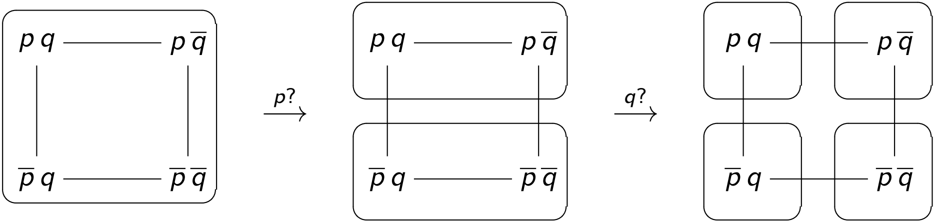
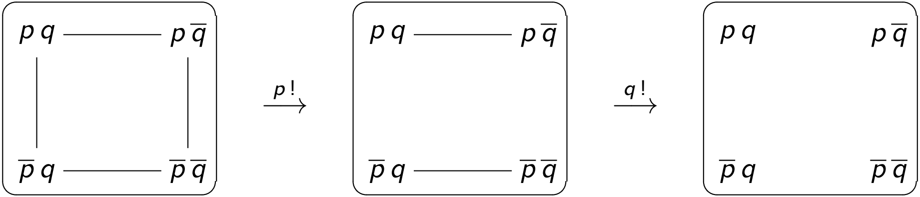


Figure: Effects of making ‘Soft’ Announcements



New Dynamic Actions of "Issue Management"

Definition (Resolution and Refinement)

An execution of the 'resolve' action ! and of the 'refine' action ?, in a model M, results in changed models $M_! = \langle W_!, \sim_!, \approx_!, V_! \rangle$ and $M_? = \langle W_?, \sim_?, \approx_?, V_? \rangle$, respectively, with:

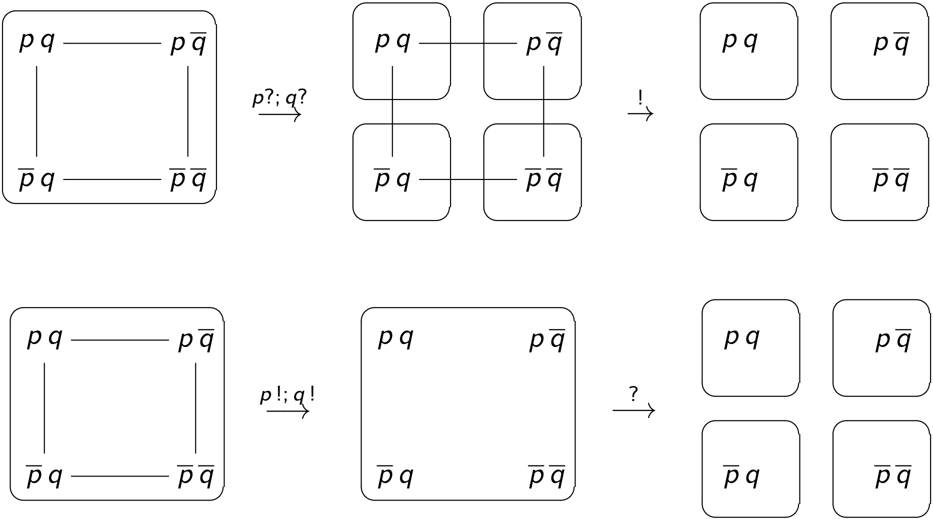
$$\begin{array}{rcl} W_? & = & W \\ \sim_? & = & \sim \\ \approx_? & = & \approx \cap \sim \\ V_? & = & V \end{array} \qquad \begin{array}{rcl} W_! & = & W \\ \sim_! & = & \sim \cap \approx \\ \approx_! & = & \approx \\ V_! & = & V \end{array}$$

$M_\# = \langle W_\#, \sim_\#, \approx_\#, V_\# \rangle$ is defined as making simultaneously:

$$\sim_\# = \approx_\# = \sim \cap \approx$$

$$W_\# = W, V_\# = V$$

Figure: Resolving and Refining Actions



Issue Management by Dynamic Questioning Actions:

;	!	?	#
!	!	#	#
?	#	?	#
#	#	#	#

(11) $\varphi!;! \neq !;\varphi!$ (12) $\varphi!;? \neq ?;\varphi!$ (13) $\varphi!;\# \neq \#;\varphi!$

(14) $\varphi?;! \neq !;\varphi!$ (15) $\varphi?;? \neq ?;\varphi!$ (16) $\varphi?;\# \neq \#;\varphi?$

(17) $\varphi?;\psi! \neq \psi!;\varphi?$

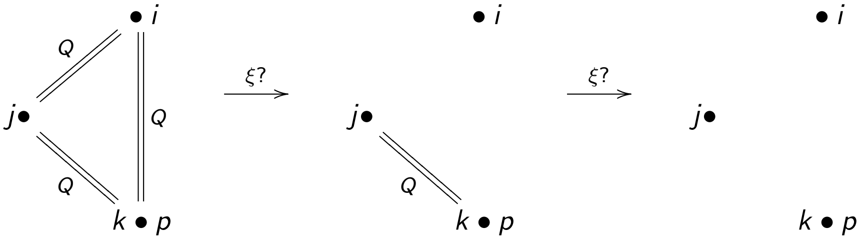
(18) $f_1?;f_2? = f_1?\cdot f_2?$ (19) $\overline{f}_1?;\overline{f}_2? \neq \overline{f}_1?\cdot\overline{f}_2?$

(20) $\varphi!;\psi? \neq \psi?;\varphi!$ (21) $\varphi!;\psi? \neq \psi?\cdot\varphi!$

(22) $pre(q)!;q \neq q;pre(q)!$ (23) $pre(q)!;q \neq pre(q)!\cdot q$

In PAL and DEL we have that $\varphi!;\varphi! \neq \varphi!$ (see Muddy Children)
Question: Is it the case that $\varphi?;\varphi? = \varphi?$ in DEL_Q?
Is the effect of a question the same if asked twice? Answer: No!

Figure: Effects of asking the same question twice



$$\xi := (\widehat{Q}i \rightarrow (j \vee k)) \wedge ((\widehat{Q}j \wedge p) \rightarrow \widehat{Q}i)$$

There are also diferences with PAL, for instance:
In PAL we have an 'action composition' principle
$\varphi!;\psi! = (\varphi \wedge [\varphi]\psi)!$.
Question: Is there an 'action contraction' principle in DEL_Q?
Answer: No!

Fact (Proper Iteration)

There is no question composition principle.

We need a logic to reason about such subtle phenomena.

Definition (Dynamic Language)

Language $\mathcal{L}_{\mathbf{DEL_Q}}(\mathtt{P},\mathtt{N})$ *is defined by adding the following clauses to the static fragment given previously in Definition 2:*

$$\cdots \mid [\varphi!]\psi \mid [\varphi?]\psi \mid [?]\varphi \mid [!]\varphi$$

These are interpreted by adding the following clauses to the recursive definition given for the static language in Definition 3:

Definition (Interpretation)

Formulas are interpreted in M *at* w *by the following clauses, where models* $M_{\varphi?}$, $M_{\varphi!}$, $M_?$ *and* $M_!$ *are as defined above:*

$$\begin{array}{lcl} M \models_w [\varphi!]\psi & \textit{iff} & M_{\varphi!} \models_w \psi, \\ M \models_w [\varphi?]\psi & \textit{iff} & M_{\varphi?} \models_w \psi, \\ M \models_w [?]\varphi & \textit{iff} & M_? \models_w \varphi \\ M \models_w [!]\varphi & \textit{iff} & M_! \models_w \varphi \end{array}$$

Our dynamic language can express useful notions:

- $[\varphi_0?] \cdots [\varphi_n?] U((\psi \to Q\psi) \wedge (\neg\psi \to Q\neg\psi))$

  This formula expresses *entailment* of questions.
- $[\varphi_0?] \cdots [\varphi_n?] \neg((\neg\psi \wedge \widehat{Q}\psi) \vee (\psi \wedge \widehat{Q}\neg\psi))$

  This formula expresses *compliance* of answers
- $K[\varphi?][\,!\,]U(K\varphi \vee K\neg\varphi)$

  This formula expresses the basic idea that gives thrust to any pattern of interrogative reasoning: the fact that the agent knows in advance that the effect of a question followed by resolution leads to knowledge

The dynamic epistemic logic of questioning based on a partition modeling (henceforth, $\mathbf{DEL_Q}$) is defined as the set of all validities:

$$\mathbf{DEL_Q} = \{\varphi \in \mathcal{L}_{\mathbf{DEL_Q}}(\mathtt{P}, \mathtt{N}) : \quad \models \varphi\}$$

Theorem (Completeness of $\mathbf{DEL_Q}$)

For every formula $\varphi \in \mathcal{L}_{\mathbf{DEL_Q}}(\mathtt{P}, \mathtt{N})$:

$$\models \varphi \quad \textit{if and only if} \quad \vdash \varphi.$$

where $\vdash$ refers to the proof system to be given below.

Proof.

Proceeds by a standard *DEL*-style translation argument. Working inside out, the reduction axioms translate dynamic formulas into corresponding static ones, in the end completeness for the static fragment is invoked. □

Reduction axioms for $\mathbf{DEL_Q}$:

1. $[q]a \leftrightarrow a$ (*Questioning & Atoms*),
2. $[q]\neg\psi \leftrightarrow \neg[q]\psi$ (*Questioning & Negation*),
3. $[q](\psi \wedge \chi) \leftrightarrow [q]\psi \wedge [q]\chi$ (*Questioning & Conjunction*),
4. $[q]U\psi \leftrightarrow U[q]\psi$ (*Questioning & Universal*),

5. $[\varphi?]K\psi \leftrightarrow K[\varphi?]\psi$ (*Asking & Knowledge*),
6. $[\varphi?]Q\psi \leftrightarrow (\varphi \wedge Q(\varphi \rightarrow [\varphi?]\psi)) \vee (\neg\varphi \wedge Q(\neg\varphi \rightarrow [\varphi?]\psi))$, (*Asking & Partition*)
7. $[\varphi?]R\psi \leftrightarrow (\varphi \wedge R(\varphi \rightarrow [\varphi?]\psi)) \vee (\neg\varphi \wedge R(\neg\varphi \rightarrow [\varphi?]\psi))$, (*Asking & Intersection*)

8. $[\,!\,]K\varphi \leftrightarrow R[\,!\,]\varphi$ (*Resolving & Knowledge*),
9. $[\,!\,]Q\varphi \leftrightarrow Q[\,!\,]\varphi$ (*Resolving & Partition*),
10. $[\,!\,]R\varphi \leftrightarrow R[\,!\,]\varphi$ (*Resolving & Intersection*),

11. $[\varphi!]K\psi \leftrightarrow (\varphi \wedge K(\varphi \to [\varphi!]\psi)) \vee (\neg\varphi \wedge K(\neg\varphi \to [\varphi!]\psi))$ (*Announcement & Knowledge*),
12. $[\varphi!]R\psi \leftrightarrow (\varphi \wedge R(\varphi \to [\varphi!]\psi)) \vee (\neg\varphi \wedge R(\neg\varphi \to [\varphi!]\psi))$ (*Announcement & Intersection*),
13. $[\varphi!]Q\psi \leftrightarrow Q[\varphi!]\psi$ (*Announcement & Partition*),

14. $[\,?\,]K\varphi \leftrightarrow K[\,?\,]\varphi$ (*Refining & Knowledge*),
15. $[\,?\,]R\varphi \leftrightarrow R[\,?\,]\varphi$ (*Refining & Intersection*),
16. $[\,?\,]Q\varphi \leftrightarrow R[\,?\,]\varphi$ (*Refining & Partition*).

We discuss two cases that are interesting as they go beyond mere commutation of operators, and illustrative for the whole enterprise.

(*Asking & Partition*) explains how questions refine a partition:

$$[\varphi?]Q\psi \leftrightarrow (\varphi \wedge Q(\varphi \rightarrow [\varphi?]\psi)) \vee (\neg\varphi \wedge Q(\neg\varphi \rightarrow [\varphi?]\psi))$$

(*Resolving & Knowledge*) shows how resolution changes knowledge (making crucial use of our intersection modality):

$$[\,!\,]K\varphi \leftrightarrow R[\,!\,]\varphi$$

(*Resolving & Knowledge*) shows how resolution changes knowledge (making crucial use of our intersection modality):

$$[\,!\,]K\varphi \leftrightarrow R[\,!\,]\varphi$$

Proof.

Let $M \models_w [\,!\,]K\varphi$. Then we have equivalently, $M_! \models_w K\varphi$ from this we get $\forall v \in W_! : w \sim_! v$ implies $M_! \models_v \varphi$.
As $\sim_! = \sim \cap \approx$, we can obtain equivalently
$\forall v \in W : w\,(\sim \cap \approx)\,v$ implies $M_! \models_v \varphi$,
finally, from this we equivalently get $M \models_w R[\,!\,]\varphi$, as desired. $\square$

Theorem (Multi-Agent $\mathbf{DEL_Q}$ Completeness)

For every formula $\varphi \in \mathcal{L}_{\mathbf{DEL_Q}}(\mathtt{P}, \mathtt{N}, \mathtt{A})$*:*

$$\models \varphi \quad \textit{if and only if} \quad \vdash \varphi.$$

where $\vdash$ refers to a proof system extended with axioms for the multi-agent case.

Proof.

Proceeds as before by a standard *DEL*-style translation argument. The only difference now is that the language contains modalities for each of the agents. $\square$

- So far we have shown that we can give a logic of questions in standard *DEL* style.

- But our analysis really shows its power (compared with alternative approaches) in the following two extensions:
  - Multi-Agent Scenarios
  - Protocols

Multi-Agent Questions

Preconditions (presuppositions) for multi-agent questions are complex and context-dependent entities:

1. $\langle \varphi ? \rangle_b \psi$ ("b asks φ"): $\neg K_b \varphi \wedge \neg K_b \neg \varphi$ (Questioner must not know the answer to the question she asks)
2. $\langle \varphi ? \rangle_a^b \psi$ ("b asks φ to a"): $\widehat{K}_b (K_a \varphi \vee K_a \neg \varphi)$ (Questioner must consider it possible that the questionee knows the answer)
3. Luxuriant variety of other types of questions: rhetorical, knowledgeable, socratic, suggestive, awareing etc.

General pattern:

$$\frac{\begin{array}{c}\text{Preconditions Announcement} \quad + \\ \text{Refinement of Issue Relation}\end{array}}{\text{Dynamic Questioning Actions}}$$

Crucial difference for multi-agent case: order **is** important!

$$pre(\langle\varphi?\rangle_a^b)!; \langle\varphi?\rangle_a^b \neq \langle\varphi?\rangle_a^b; pre(\langle\varphi?\rangle_a^b)!$$

$$pre(\langle\varphi?\rangle_a^b)!; \langle\varphi?\rangle_a^b \neq pre(\langle\varphi?\rangle_a^b)! \cdot \langle\varphi?\rangle_a^b$$

$$\langle\varphi?\rangle_a^b; pre(\langle\varphi?\rangle_a^b)! \neq \langle\varphi?\rangle_a^b \cdot pre(\langle\varphi?\rangle_a^b)!$$

We have to handle **simultaneously** two components:

- Complex pressupositions for very general (even private) multi-agent questions:
  - Handled by the general DEL mechanism for announcements.
- Complex transformations of the issue relations for very general (even private) multi-agent questions:
  - Handled well by simple refinement for public questions, but in order to handle private question we need more general product update mechanism on suitable event models.

Definition (Interpretation)

Formulas are interpreted in M at w by the following clauses, where models $M_{\varphi?}$, $M_{\varphi!}$, $M_?$ and $M_!$ are as defined above for multi agent:

$$\begin{array}{lll}
M \models_w [\varphi?]^b_a\psi & \textit{iff} & M \models_w \mathtt{pre}(\varphi?\,^b_a) \textit{ implies } M_{\varphi?^b_a\cdot\mathtt{pre}(\varphi?\,^b_a)!} \models_w \psi, \\
M \models_w [\varphi!]^b_a\psi & \textit{iff} & M \models_w \mathtt{pre}(\varphi!^b_a) \textit{ implies } M_{\varphi!^b_a\cdot\mathtt{pre}(\varphi!\,^b_a)!} \models_w \psi, \\
M \models_w [?]\varphi & \textit{iff} & M_? \models_w \varphi, \\
M \models_w [\,!\,]\varphi & \textit{iff} & M_! \models_w \varphi.
\end{array}$$

Theorem (Multi-Action $\mathbf{DELQ_M}$ Completeness)

For every $\varphi \in \mathcal{L}_{\mathbf{DELQ_M}}(\mathtt{P}, \mathtt{N}, \mathtt{A})$:

$$\models \varphi \quad \text{if and only if} \quad \vdash \varphi.$$

where $\vdash$ refers to the proof system given below:

Reduction axioms for $\mathbf{DELQ_M}$:

1. (*Questioning & Atoms*): $[q]t \leftrightarrow t$,
2. (*Questioning & Negation*): $[q]\neg\psi \leftrightarrow \neg[q]\psi$,
3. (*Questioning & Conjunction*): $[q](\psi \wedge \chi) \leftrightarrow [q]\psi \wedge [q]\chi$,
4. (*Questioning & Universal*): $[q]U\psi \leftrightarrow U[q]\psi$,

5. (*Asking & Knowledge*), where $\chi = \mathtt{pre}(\varphi?\,_a^b)$:
   $[\varphi?]_a^b K_c\psi \leftrightarrow (\chi \wedge K_c(\chi \to [\varphi?]_a^b\psi)) \vee (\neg\chi \wedge K_c(\neg\chi \to [\varphi?]_a^b\psi))$,
6. (*Asking & Partition*): $[\varphi?]_a^b Q_a\psi \leftrightarrow$
   $(\varphi \wedge Q_a(\varphi \to [\varphi?]_a^b\psi)) \vee (\neg\varphi \wedge Q_a(\neg\varphi \to [\varphi?]_a^b\psi))$,
7. (*Ask&Intrsetion*): $[\varphi?]_a^b R_c\psi \leftrightarrow \bigvee_i \{\chi_i \wedge R_c(\chi_i \to [\varphi?]_a^b\psi)\}$,

$\chi_i \in \{\mathtt{pre}(\varphi?\,_a^b) \wedge \varphi, \neg\mathtt{pre}(\varphi?\,_a^b) \wedge \varphi, \mathtt{pre}(\varphi?\,_a^b) \wedge \overline{\varphi}, \neg\mathtt{pre}(\varphi?\,_a^b) \wedge \overline{\varphi}\}$

11. (*Announcement & Knowledge*): $[\varphi!]_a^b K_c\psi \leftrightarrow \bigvee_i \{\chi_i \wedge K_c(\chi_i \to [\varphi!]_a^b\psi)\}$,
12. (*Announcement & Partition*): $[\varphi!]_a^b Q_c\psi \leftrightarrow Q_c[\varphi!]_a^b\psi$,
13. (*Ann&Intrsction*): $[\varphi!]_a^b R_c\psi \leftrightarrow \bigvee_i \{\chi_i \wedge R_c(\chi_i \to [\varphi!]_a^b\psi)\}$,

$$\chi_i \in \{\mathtt{pre}(\varphi!\,{}_a^b) \wedge \varphi, \neg\mathtt{pre}(\varphi!\,{}_a^b) \wedge \varphi, \mathtt{pre}(\varphi!\,{}_a^b) \wedge \overline{\varphi}, \neg\mathtt{pre}(\varphi!\,{}_a^b) \wedge \overline{\varphi}\}$$

14. (*Refining & Knowledge*): $[\,?\,]K_c\varphi \leftrightarrow K_c[\,?\,]\varphi$,
15. (*Refining & Intersection*): $[\,?\,]R_c\varphi \leftrightarrow R_c[\,?\,]\varphi$,
16. (*Refining & Partition*): $[\,?\,]Q_c\varphi \leftrightarrow R_c[\,?\,]\varphi$.

8. (*Resolving & Knowledge*): $[\,!\,]K_c\varphi \leftrightarrow R_c[\,!\,]\varphi$,
9. (*Resolving & Partition*): $[\,!\,]Q_c\varphi \leftrightarrow Q_c[\,!\,]\varphi$,
10. (*Resolving & Intersection*): $[\,!\,]R_c\varphi \leftrightarrow R_c[\,!\,]\varphi$,

DELQ with Private Questions

Definition (Questioning Action Model)

An epistemic-issue event model *is a structure* $Q = \langle E, \overset{a}{\sim}, \overset{a}{\approx}, \mathtt{pre} \rangle$*:*

- *E is a set of abstract epistemic events (or epistemic actions),*
- *$\overset{a}{\sim}$ is a family of equivalence relations on E (indistinguishability),*
- *$\overset{a}{\approx}$ is a family of equivalence relations on E (issue equivalence),*
- $\mathtt{pre} : E \to \wp(\mathcal{L}_{\mathbf{DEL_Q}}(\mathtt{P}, \mathtt{N}, \mathtt{A}))$ *is a precondition function mapping events into sets of formulas (preconditions for action execution).*

Definition (Question-Adequate Model)

An event model is **adequate** *for questions under the following conditions:*

1. $\forall Q_i \in Q, \forall q_i \in Q_i : q_i \in Q_i \Rightarrow \exists e \in E \wedge e = q_i$,
   (every possible answer to a modeled questions is modeled)
2. $\forall a \in A, \forall e, e' \in E : (e, e') \in \overset{a}{\sim}$,
   (all modeled agents are blissfully ignorant in the model)
3. *(indistinguishable questions have issue-equivalent answers)*
4. $\forall w \in W, \forall q \in Q_i \in Q : (w, q) \in W_\times \Leftrightarrow M \models_w q$.
   (every action, i.e. answer, is executable only when it is true)

Definition (Question Product Update)

Given epistemic and action issue models $M = \langle W, \overset{a}{\sim}, \overset{a}{\approx}, V\rangle$ *and* $Q = \langle E, \overset{a}{\sim}, \overset{a}{\approx}, \texttt{pre}\rangle$, *the* **product uqdate model** *is defined as* $M \times Q = \langle W_\times, \overset{a}{\sim}_\times, \overset{a}{\approx}_\times, V_\times\rangle$ *where:*

$$W_\times = \{(w,q) \mid w \in W, q \in E, w \in \texttt{pre}(q)\}$$

$$\overset{a}{\sim}_\times = \{((w,q),(w',q')) \mid w \overset{a}{\sim} w', q \overset{a}{\sim} q', \}$$

$$\overset{a}{\approx}_\times = \{((w,q),(w',q')) \mid w \overset{a}{\approx} w', q \overset{a}{\approx} q', \}$$

$$V_\times = V, W_\times \supseteq W^*_\times = \{(w,q) \mid w \in W^*, q \in E^*\}$$

Definition (Language)

The language $\mathcal{L}_{\mathbf{DELQ}}(\mathtt{P}, \mathtt{N}, \mathtt{A}, \mathtt{Z})$, *with* $p \in \mathtt{P}$, $i \in \mathtt{N}$, $a \in \mathtt{A}$ *and questioning actions* $\zeta \in \mathtt{Z}$*:*

$$i \mid p \mid \bot \mid \neg\varphi \mid (\varphi \wedge \psi) \mid U\varphi \mid K_a\varphi \mid Q_a\varphi \mid R_a\varphi \mid [\zeta]\varphi \mid [\,!\,]\varphi$$

here ζ *is an adequate questioning model, (1) has a finite domain, & (2) every precondition has priority in the inductive hierarchy.*

Definition (Interpretation)

Formulas are interpreted as follows:

$$\begin{array}{rcl} M \models_w [\,!\,]\varphi & \textit{iff} & M_! \models_w \varphi, \\ M \models_w [\zeta]\varphi & \textit{iff} & (M, w)[\![\zeta]\!](M', w') \textit{ implies } M' \models_w \varphi, \\ (M, w)[\![\zeta]\!](M', w') & \textit{iff} & M \models_w \mathtt{pre}(\zeta) \textit{ and } (M', w') = (M \times \zeta). \end{array}$$

Theorem (DELQ Completeness)

For every $\varphi \in \mathcal{L}_{\mathbf{DELQ}}(\mathtt{P}, \mathtt{N}, \mathtt{A}, \mathtt{Z})$*:*

$$\models \varphi \quad \textit{if and only if} \quad \vdash \varphi.$$

where $\vdash$ refers to the proof system to be given below.

Proof.

Proceeds again by a standard *DEL*-style translation argument. $\square$

Reduction axioms for **DELQ**:

1. (*Questioning & Atoms*): $[Q]t \leftrightarrow (\mathtt{pre}(Q) \to t)$,
2. (*Questioning & Negation*): $[Q]\neg\psi \leftrightarrow (\mathtt{pre}(Q) \to \neg[Q]\psi)$,
3. (*Questioning & Conjunction*): $[Q](\psi \wedge \chi) \leftrightarrow [Q]\psi \wedge [Q]\chi$,
4. (*Questioning & Universal*): $[Q]U\psi \leftrightarrow (\mathtt{pre}(Q) \to U[Q]\psi)$,

5. (*Asking & Knowledge*): $[E, q]K_a\varphi \leftrightarrow (\mathtt{pre}(Q) \to K_a[E, q]\varphi)$,
6. (*Asking & Partition*):
   $[E, q]Q_a\varphi \leftrightarrow (\mathtt{pre}(Q) \to \bigwedge_{q \overset{a}{\approx} q'} Q_a[E, q']\varphi)$,
7. (*Asking & Intersection*):
   $[E, q]R_a\varphi \leftrightarrow (\mathtt{pre}(Q) \to \bigwedge_{q \overset{a}{\approx} q'} R_a[E, q']\varphi)$,

8. (*Resolving & Knowledge*): $[\,!\,]K_a\varphi \leftrightarrow R_a[\,!\,]\varphi$,
9. (*Resolving & Partition*): $[\,!\,]Q_a\varphi \leftrightarrow Q_a[\,!\,]\varphi$,
10. (*Resolving & Intersection*): $[\,!\,]R_a\varphi \leftrightarrow R_a[\,!\,]\varphi$.

Protocols & Procedural Constraints

- Questions are usualy part of inquiry scenarios subject to various procedural restrictions.
- These can also be modeled by recent developments from PAL/DEL: Protocols. (van Benthem, Gerbrandy, Hoshi, Pacuit 2009)

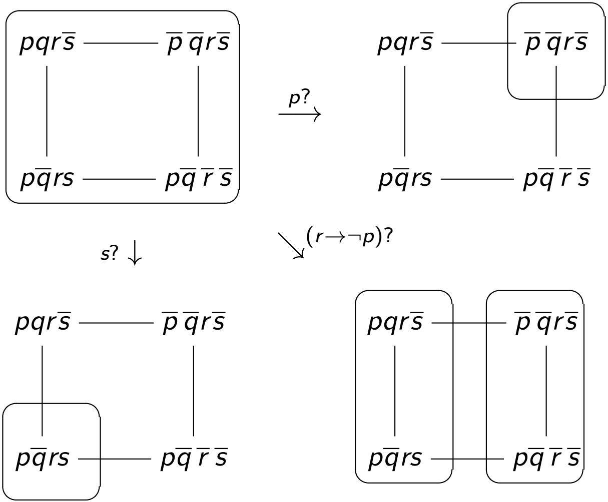


Figure: Experiments are more efficient than atomic questioning

$$Q_1 = \{p?, q?, p?!, q?!, p?!q?, q?!p?, p?!q?!, q?!p?!\}$$
$$Q_2 = \{p?, q?, p?q?, q?p?, p?q?!, q?p?!\}$$

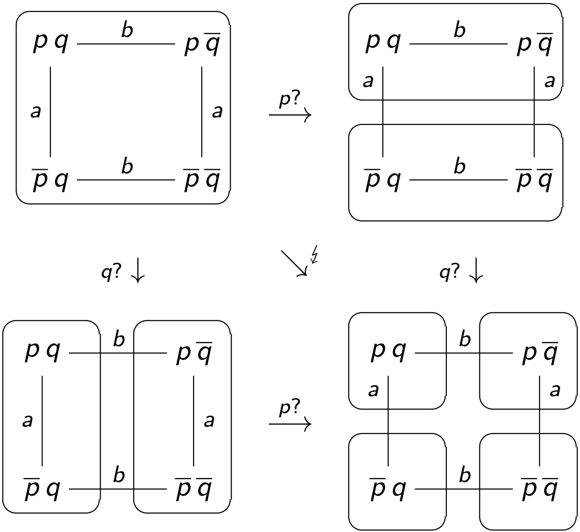


Figure: Fairness of cooperative experimental procedures

$$Q_1 = \{p?, q?, p?!, q?!, p?!q?, q?!p?, p?!q?!, q?!p?!\}$$

$$\mathrm{Fr}(M, Q_1) \models_{p?!} UK_a(\rho) \wedge U\neg K_b(\rho)$$

$$\mathrm{Fr}(M, Q_1) \models_{q?!} UK_b(\rho) \wedge U\neg K_a(\rho)$$

$$Q_2 = \{p?, q?, p?q?, q?p?, p?q?!, q?p?!\}$$

$$\mathrm{Fr}(M, Q_2) \models UK_i(\rho) \leftrightarrow UK_j(\rho)$$

$$\rho := (p \wedge q) \vee (\overline{p} \wedge q) \vee (p \wedge \overline{q}) \vee (\overline{p} \wedge \overline{q})$$

Sample axioms for $TDEL_Q$:

Questions & Partition:

$$\langle\varphi?\rangle Q\psi \leftrightarrow \langle\varphi?\rangle\top\wedge((\varphi\wedge Q(\varphi \to \langle\varphi?\rangle\psi))\vee(\neg\varphi\wedge Q(\neg\varphi \to \langle\varphi?\rangle\psi)))$$

Resolution & Knowledge:

$$\langle !\rangle K\varphi \leftrightarrow \langle !\rangle\top \wedge R\langle !\rangle\varphi$$

Refinement & Issue:

$$\langle ?\rangle Q\varphi \leftrightarrow \langle ?\rangle\top \wedge R\langle ?\rangle\varphi$$

Theorem (Completeness of $\mathbf{TDEL_Q}$)

For every formula $\varphi \in \mathcal{L}_{\mathbf{TDEL_Q}}(\mathtt{P}, \mathtt{N}, A)$*:*

$$\models \varphi \quad \textit{if and only if} \quad \vdash \varphi.$$

where $\vdash$ refers to a proof system extended with suitable axioms in the style of the previous samples.

Further Research Topics:

- Epistemic Games with Questions & Announcements
- Syntactic Approaches to Questioning Phenomena:
  - Inference, Questions & Awareness Promotion
  - Discovery, Inquiry, & Dynamics of Research Agendas
  - Interaction with other Epistemic & Doxastic Attitudes