Agreement Theorems from the Perspective of Dynamic-Epistemic Logic

Olivier Roy & Cedric Dégremont

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Overview

1. Introduction to Agreements Theorems (AT)
2. Models of knowledge and beliefs, common priors and common knowledge
3. Three variations on the result: static, kinematic and dynamic

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2. Models of knowledge and beliefs, common priors and common knowledge
3. Three variations on the result: static, kinematic and dynamic

Highlights:

- The received view:
  - ATs undermine the role of private information.
- The DEL point of view:
  - ATs show the importance of higher-order information.
  - ATs show how "static" conditioning is different from "real" belief dynamics.

Agreement Theorems in a nutshell

The Annals of Statistics
1976, Vol. 4, No. 6, 1236–1239

AGREEING TO DISAGREE[1]

BY ROBERT J. AUMANN

Stanford University and the Hebrew University of Jerusalem

Two people, 1 and 2, are said to have *common knowledge* of an event E if both know it, 1 knows that 2 knows it, 2 knows that 1 knows is, 1 knows that 2 knows that 1 knows it, and so on.

THEOREM. *If two people have the same priors, and their posteriors for an event A are common knowledge, then these posteriors are equal.*

If two people have the same priors, and their posteriors for a given event A are common knowledge, then these posteriors must be equal. This is so even though they may base their posteriors on quite different information. In brief, people with the same priors *cannot agree to disagree*.

We publish this observation with some diffidence, since once one has the appropriate framework, it is mathematically trivial. Intuitively, though, it is not quite obvious; and it is of some interest in areas in which people's beliefs about

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A short result goes a long way

- Original result: [Aumann, 1976].
- No trade theorems: [Milgrom and Stokey, 1982].
- "Dynamic" versions: [Geanakoplos and Polemarchakis, 1982]
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- Good survey: [Bonanno and Nehring, 1997].

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- How important is private information? Not quite...
- How strong is the common knowledge condition? Very...
- How plausible is the common prior assumption? Debated...

Conclusions, the point of view of DEL.

Theorem

If two people have the same priors, and their posteriors for an event A are common knowledge, then these posterior are the same.

- The key is higher-order information.
- One should distinguish information kinematics *vs* information dynamics, belief conditioning *vs* belief update.

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- $\sim_i$ is an *epistemic accessibility equivalence relation*. We write $[w]_i$ for $\{w' : w \sim_i w'\}$.

See: [Board, 2004, Baltag and Smets, 2008, van Benthem,]

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- Common knowledge: $w \models CK_I\varphi$ iff $w' \models E_I\varphi$ and $w' \models E_I E_I\varphi$ and...
- Beliefs: $w \models B_i^{\psi}\varphi$ iff $w' \models \varphi$ for all w' in $max_{\leq_i}([w]_i \cap ||\psi||)$. We write $B_i\varphi$ for $B_i^{\top}\varphi$.

Theorem (Static agreement)

For any epistemic-doxastic model $\mathbb{M}$ with common priors, for all w we have that

$$w \not\models CK_I(B_i(E) \wedge \neg B_j(E))$$

where $E \subseteq W$.

The key property: Sure-thing principle

> *If, first, you would believe, conditional on the fact that it is cloudy, that it will rain and,*

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> *If, first, you would believe, conditional on the fact that it is cloudy, that it will rain and, second, you would believe, conditional on the fact that it is not cloudy, that it will rain, then you unconditionally believe that it will rain.*

[Savage, 1954, Bacharach, 1985]

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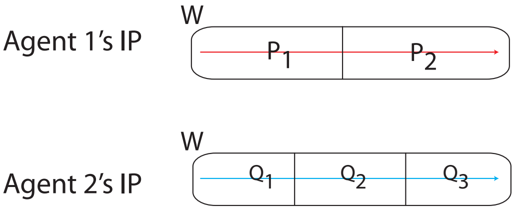

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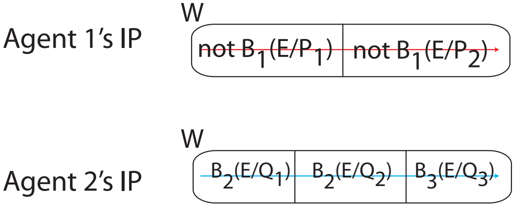

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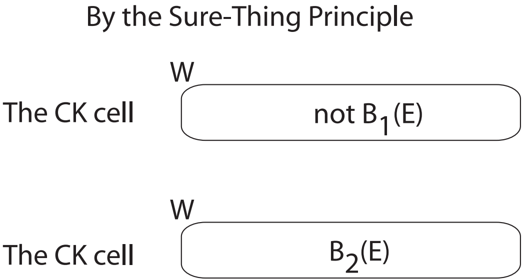

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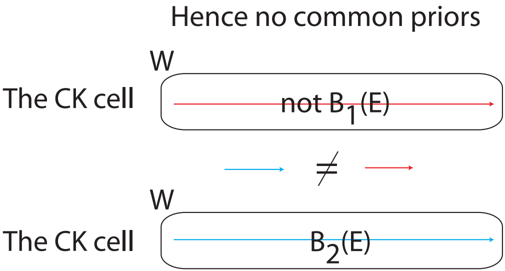

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The cornerstone is higher-order information.

Question:

- How can CK obtain with respect to each others' beliefs?
  - "Dialogues" or repeated announcements. [Geanakoplos and Polemarchakis, 1982, Bacharach, 1985]

Towards kinematic agreement

Definition

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- $\sim_{0,i}$ is a given epistemic accessibility relation.
- for all $w \in W$:

$$[w]_{n+1,i} \;=\; [w]_{n,i} \cap \begin{cases} B_n(A) & \text{if } w \models B_{n,j}(A) \\ \neg B_{n,j}(A) & \textit{otherwise}. \end{cases}$$

with $B_{n,j}(A) = \{w' : max_{\leq_j}[w']_{n,j} \subseteq A\}$.

Intuition: $B_{n+1,i}\varphi \Leftrightarrow B_{n,i}^{B_{n,j}\varphi}\varphi$.

Lemma (Fixed-point)

Every kinematic dialogue about A has a fixed-point, i.e. there is a n^* such that

$$[w]_{n^*,i} = [w]_{n^*+1,i}$$

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Theorem (Kinematic agreement)

For any kinematic dialogue about A, if there is common priors then at the fixed-point either all agents believe that A or they all don't believe that A.

Lesson:

- Common knowledge arise from "dialogues"...

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Warnings from DEL:

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- Common knowledge arise from "dialogues"...

Warnings from DEL:

- This is only a kinematic (i.e. conditioning) dialogue: the truth of A is fixed during the process.
- In general, this is not the case. A might be about the agents' information.
- Another way to look at it:

$$\text{kinematic agreement} = \text{“virtual” agreement}$$

Towards dynamic agreement

Definition

A *dynamic* dialogue about A is sequence of epistemic-doxastic pointed models $\{(\mathbb{M}_n, w)\}_{n\in\mathbb{N}}$ such that:

- $\mathbb{M}_0$ is a given epistemic-doxastic model.
- $\mathbb{M}_{n+1} = \langle W_{n+1}, I, \leq_{n+1,i}, \sim_{n+1,i}\rangle$ with
  - $W_{n+1} = \{w' \in W_n : w' \models B_n(A_n)\}$ with:
    - $A_n = \{w' \in W_n : w' \models A\}$
    - $B_n(A_n)$ is $B_{n,i}(A_n) \wedge B_{n,j}(A_n)$ if $w \models B_{n,i}(A_n) \wedge B_{n,j}(A_n)$, etc...
  - $\leq_{n+1,i}$ $\sim_{n+1,i}$ are the restrictions of $\leq_{n,i}$ and $\sim_{n,i}$ to W_{n+1}.

Intuition: $B_{n+1,i}\varphi \Leftrightarrow [B_n\varphi!]B_{n,i}\varphi$

Lemma (Fixed-point)

Every dynamic dialogue about A has a fixed-point, i.e. there is a n^* such that:

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The posteriors beliefs at the fixed-point of a dynamic dialogue are common knowledge.

Theorem (Dynamic agreement)

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Theorem (Dynamic agreement)

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But...

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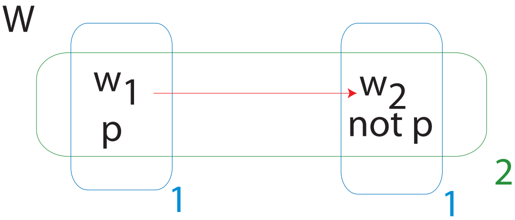

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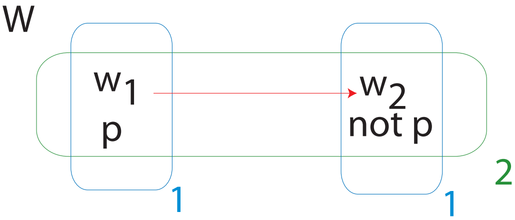

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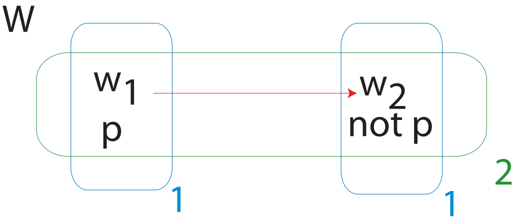

- Let $A = p \wedge \neg B_2 p$
- At the fixed points for the kinematic and the dynamic dialogue about A, we have that $[w_1]_{n^*,1} = [w_1]_{n^*,2} = \{w_1\}$

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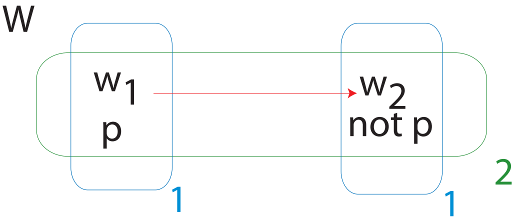

- Let $A = p \wedge \neg B_2 p$
- At the fixed points for the kinematic and the dynamic dialogue about A, we have that $[w_1]_{n^*,1} = [w_1]_{n^*,2} = \{w_1\}$
- Kinematic beliefs: $w \models B_{n^*,i}(p \wedge \neg B_{0,2} p)$, for $i = 1, 2$.
- Dynamic beliefs: $w \models \neg B_{n^*,i}(p \wedge \neg B_{n^*,2} p)$, for $i = 1, 2$.

- Agreements theorems:
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  - Undermine the role of private information?
  - A better way to look at it ("DEL methodology"):
    - Rest on higher-order information and its role in interactive reasoning.
    - Highlight the difference between belief kinematics and belief dynamics.
- (Hopefully not so distant) future work :
  - General (countable) case?
  - Announcements of reasons and not only of opinions?
  - Relaxing the common prior assumption? Agreements on everything?

Aumann, R. (1976).
Agreeing to disagree.
The Annals of Statistics, 4(6):1236–1239.

Bacharach, M. (1985).
Some extensions of a claim of aumann in an axiomatic model of knowledge.
Journal of Economic Theory, 37(1):167–190.

Baltag, A. and Smets, S. (2008).
A qualitative theory of dynamic interactive belief revision.
In Bonanno, G., van der Hoek, W., and Wooldridge, M., editors, *Logic and the Foundation of Game and Decision Theory (LOFT7)*, volume 3 of *Texts in Logic and Games*, pages 13–60. Amsterdam University Press.

Board, O. (2004).
Dynamic Interactive Epistemology.
Games and Economic Behavior, 49:49–80.

Bonanno, G. and Nehring, K. (1997).
Agreeing to disagree: a survey.
Some of the material in this paper was published in [Bonanno and Nehring, 1999].

Bonanno, G. and Nehring, K. (1999).
How to make sense of the common prior assumption under incomplete information.
International Journal of Game Theory, 28(03):409–434.

Cave, J. A. K. (1983).
Learning to agree.
Economics Letters, 12(2):147–152.

Geanakoplos, J. and Polemarchakis, H. M. (1982).
We can't disagree forever.

Cowles Foundation Discussion Papers 639, Cowles Foundation, Yale University.

Milgrom, P. and Stokey, N. (1982).
Information, Trade, and Common Knowledge.
Journal of Economic Theory, 26:17–27.

Parikh, R. and Krasucki, P. (1990).
Communication, consensus, and knowledge.
Journal of Economic Theory, 52(1):178–189.

Savage, L. (1954).
The Foundations of Statistics.
Dover Publications, Inc., New York.

van Benthem, J.
Logical dynamics of information and interaction.
Manuscript.