From game logics to logic games: a strategy perspective

(ongoing work)

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objectives

- Parikh's Game logic gives us a framework to talk about both games and strategies.
- How does this framework relate to evaluation games for different logics?

outline

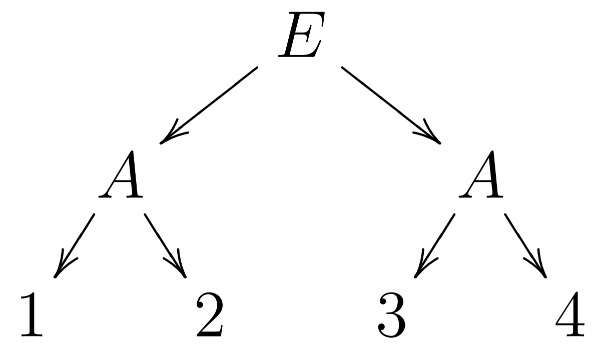
- game logic
- incorporating strategies
- evaluation games
- some representation result(s)

game logic

games as forcing relations:

$w\rho_G^i X$ player i has a strategy for playing game *G* from state w onwards, whose resulting states are always in the set X.

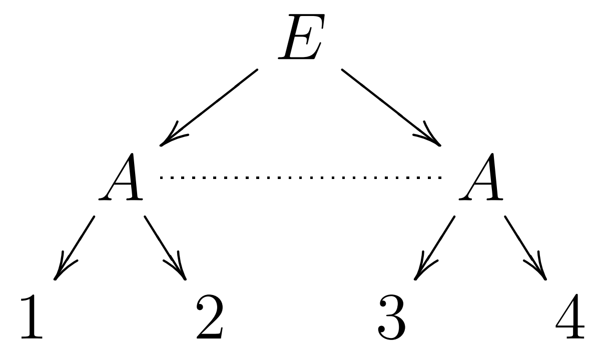
game logic



E's powers : {1, 2}, {3, 4}.

A's powers : {1, 3}, {1, 4}, {2, 3}, {2, 4}.

game logic



E's powers : {1, 2}, {3, 4}.

A's powers : {1, 3}, {2, 4}.

Conditions on Forcing Relations:

- Monotonicity: If $s\rho_G^i X$ and $X \subseteq X'$, then $s\rho_G^i\ X'$.
- Consistency: If $s\rho_G^E Y$ and $s\rho_G^A Z$, then Y and Z overlap.
- Determinacy: If it is not the case that s ρ_G^E X, then, s ρ_G^A S - X, and the same for A vis-a-vis E, where S denotes the total set of states.

game logic

composite game structures:

Choice ($G \cup G'$),

Dual (G^d),

Sequential composition ($G; G'$)

game logic

determined version:

$w\rho^E_{G\cup G'}X$	iff	$w\rho^E_G X$ or $w\rho^E_{G'}X$
$w\rho^E_{G^d}X$	iff	it is not the case that $w\rho^E_G X^{\mathcal{C}}$
$w\rho^E_{G;G'}X$	iff	$\exists Z$: $w\rho^i_G Z$ and for all $z \in Z$, $z\rho^i_{G'}X$
$w\rho^E_{\varphi?}X$	iff	$w \in [[\varphi]]$ and $w \in X$

game logic

non-determined version:

$w\rho^E_{G\cup G'}X$	iff	$w\rho^E_G X$ or $w\rho^E_{G'}X$
$w\rho^A_{G\cup G'}X$	iff	$w\rho^A_G X$ and $w\rho^A_{G'}X$
$w\rho^E_{G^d}X$	iff	$w\rho^A_G X$
$w\rho^A_{G^d}X$	iff	$w\rho^E_G X$
$w\rho^i_{G;G'}X$	iff	$\exists Z : w\rho^i_G Z$ and for all $z \in Z$, $z\rho^i_{G'}X$
$w\rho^E_{\varphi?}X$	iff	$w \in [[\varphi]]$ and $w \in X$
$w\rho^A_{\varphi?}X$	iff	$w \notin [[\varphi]]$ and $w \in X$

game logic

language:

$\gamma := g \mid \varphi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^d$

$\phi := \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\gamma\rangle\varphi$

Game Model: $\mathcal{M} = (W, \{\rho_g \mid g \in \Gamma\}, V)$, $\rho_g \subseteq W \times \mathcal{P}(W)$

Semantics:

$\mathcal{M}, w \models \langle\gamma\rangle\varphi$ iff there exists $X : w\rho_\gamma X$ and

$\forall x \in X : \mathcal{M}, x \models \varphi$

game logic

some valid statements and rules:

- $\langle\gamma \cup \gamma'\rangle\varphi \leftrightarrow \langle\gamma\rangle\varphi \vee \langle\gamma'\rangle\varphi$
- $\langle\gamma^d\rangle\varphi \leftrightarrow \neg\langle\gamma\rangle\neg\varphi$
- $\langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi$
- $\langle\varphi?\rangle\psi \leftrightarrow (\varphi \wedge \psi)$
- if $\vdash \varphi \rightarrow \psi$ then $\vdash \langle\gamma\rangle\varphi \rightarrow \langle\gamma\rangle\psi$

game logic

language:

$\gamma := g \mid \phi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^d$

$\phi := \bot \mid p \mid \neg\phi \mid \phi \vee \phi \mid \langle \gamma, i \rangle \phi$

Game Model: $\mathcal{M} = (W, \{\rho_g^i \mid g \in \Gamma\}, V)$, $\rho_g \subseteq W \times \mathcal{P}(W)$

Semantics:

$\mathcal{M}, w \models \langle \gamma, i \rangle \phi$ iff there exists $X : w\rho_\gamma^i X$ and $\forall x \in X : \mathcal{M}, x \models \varphi$

game logic

some valid statements and rules:

- $\langle \gamma \cup \gamma', E \rangle \varphi \leftrightarrow \langle \gamma, E \rangle \varphi \vee \langle \gamma', E \rangle \varphi$
- $\langle \gamma \cup \gamma', A \rangle \varphi \leftrightarrow \langle \gamma, A \rangle \varphi \wedge \langle \gamma', A \rangle \varphi$
- $\langle \gamma^d, E \rangle \varphi \leftrightarrow \langle \gamma, A \rangle \varphi$
- $\langle \varphi ?, E \rangle \psi \leftrightarrow (\varphi \wedge \psi)$
- $\langle \varphi ?, A \rangle \psi \leftrightarrow (\neg \varphi \wedge \psi)$
- if $\vdash \varphi \rightarrow \psi$ then $\vdash \langle \gamma, i \rangle \varphi \rightarrow \langle \gamma, i \rangle \psi$

game logic

- input-output behavior
- generic games on game boards
- quantification over strategies to achieve something
- winning strategy vs. φ-strategy
- uniform study

incorporating strategies

- winning strategies
- best response strategy
- strategies to achieve something

$$\vdots$$

incorporating strategies

games as forcing relations:

$w\rho^i_G X$ player i has a strategy for playing game G from state w onwards, whose resulting states are always in the set X.

strategy in game as forcing relations:

$w\rho^i_{\langle G,S\rangle} X$ by following the strategy S in the game G from state w onwards, the resulting final states reached by the player i are always in the set X, whatever the other player chooses to do.

incorporating strategies

conditions on these strategy-forcing relations:

- monotonicity: if $w\rho^i_{\langle G,S\rangle}X$ and $X \subseteq X'$, then $w\rho^i_{\langle G,S\rangle}\ X'$.
- consistency: if $w\rho^E_{\langle G,S\rangle}Y$ and $w\rho^A_{\langle G,S'\rangle}Z$, then Y and Z overlap.
- subset: $\rho^i_{\langle G,S\rangle} \subseteq \rho^i_G$

incorporating strategies

composite game-strategy structures:

Choice ($G \cup G', S \cup S'$),

Dual (G^d, S),

Sequential composition ($G; G', S; S'$),

Test games ($\varphi?, \top$), ($\varphi?, \bot$)

incorporating strategies

$w\rho^E_{\langle G\cup G', S\cup S'\rangle}$ X	iff	$w\rho^E_{\langle G,S\rangle}$ X or $w\rho^E_{\langle G',S'\rangle}X$
$w\rho^A_{\langle G\cup G', S\cup S'\rangle}$ X	iff	$w\rho^A_{\langle G,S\rangle}X$ and $w\rho^A_{\langle G',S'\rangle}X$

$w\rho^E_{\langle G^d,S\rangle}X$	iff	$w\rho^A_{\langle G,S\rangle}X$
$w\rho^A_{\langle G^d,S\rangle}X$	iff	$w\rho^E_{\langle G,S\rangle}X$

$w\rho^i_{\langle G;G',S;S'\rangle}X$	iff	$\exists Z$: $w\rho^i_{\langle G,S\rangle}Z$ and for all $z \in Z$, $z\rho^i_{\langle G',S'\rangle}X$

$w\rho^E_{\langle \varphi ?,\top\rangle}X$	iff	$w\rho^A_{\langle \varphi ?,\bot\rangle}\overline{X}$	iff	$w\rho^E_{\varphi ?}X$	iff	$w \in [[\varphi]]$ and $w \in X$
$w\rho^A_{\langle \varphi ?,\top\rangle}X$	iff	$w\rho^E_{\langle \varphi ?,\bot\rangle}\overline{X}$	iff	$w\rho^A_{\varphi ?}X$	iff	$w \notin [[\varphi]]$ or $w \in X$

incorporating strategies

Language:

$\gamma := g \mid \varphi? \mid \gamma;\gamma \mid \gamma \cup \gamma \mid \gamma^d$

$\sigma := s \mid \top \mid \bot \mid \sigma;\sigma \mid \sigma \cup \sigma$

$\varphi := \bot \mid p \mid \neg\varphi \mid \varphi \vee \varphi \mid \langle\gamma\rangle\varphi \mid \langle(\gamma,\sigma),i\rangle\varphi$

In the syntax for $\langle(\gamma,\sigma),i\rangle$, we require that $\sigma \in \Sigma(\gamma)$, where, for each game γ, $\Sigma(\gamma)$ is defined inductively as follows:

$$\Sigma(g) = \Sigma$$
$$\Sigma(\phi?) = \{\top,\bot\}$$
$$\Sigma(\gamma^d) = \Sigma(\gamma)$$
$$\Sigma(\gamma;\gamma) = \{\sigma;\sigma' : \sigma \in \Sigma(\gamma) \text{ and } \sigma' \text{ in } \Sigma(\gamma')\}$$
$$\Sigma(\gamma \cup \gamma) = \{\sigma \cup \sigma' : \sigma \in \Sigma(\gamma) \text{ and } \sigma' \in \Sigma(\gamma')\ \}$$

incorporating strategies

Game Model:

$\mathcal{M} = (W, \{\rho_g \mid g \in \Gamma\}, \{\rho^i_{\langle g,s\rangle} \mid g \in \Gamma, s \in \Sigma\}, V)$,
$\rho^i_g, \rho^i_{\langle g,s\rangle} \subseteq W \times \mathcal{P}(W)$, both monotonic, with $\rho^i_g \supseteq \rho^i_{\langle g,s\rangle}$ and
$s\rho^A_g X$ iff it is not that $s\rho^E_g(W \setminus X)$.

Semantics:

$\mathcal{M}, w \models \langle(\gamma, \sigma), i\rangle\varphi$ iff there exists $X \subseteq W : w\rho^i_{\langle\gamma,\sigma\rangle}X$ and
for all $x \in X : \mathcal{M}, x \models \varphi$.

incorporating strategies

some valid statements and rules:

- $\langle(\gamma,\sigma),E\rangle\varphi \to \langle\gamma,E\rangle\varphi$
- $\langle(\gamma\cup\gamma',\sigma\cup\sigma'),E\rangle\varphi \leftrightarrow \langle(\gamma,\sigma),E\rangle\varphi \vee \langle(\gamma',\sigma'),E\rangle\varphi$
- $\langle(\gamma\cup\gamma',\sigma\cup\sigma'),A\rangle\varphi \leftrightarrow \langle(\gamma,\sigma),A\rangle\varphi \wedge \langle(\gamma',\sigma'),A\rangle\varphi$
- $\langle(\gamma^d,\sigma),E\rangle\varphi \leftrightarrow \langle(\gamma,\sigma),A\rangle\varphi$
- $\langle(\varphi?,\top),E\rangle\psi \leftrightarrow (\varphi\wedge\psi)$
- $\langle(\varphi?,\top),A\rangle\psi \leftrightarrow (\neg\varphi\vee\psi)$
- if $\vdash \varphi\to\psi$ then $\vdash \langle(\gamma,\sigma),i\rangle\varphi \to \langle(\gamma,\sigma),i\rangle\psi$

questions and comments

- using a '$\perp$'-like syntax for 'no available strategy' (as in the test games), we can do away with the game formulas, where $\langle(\gamma, \perp), E\rangle\varphi$ will be equivalent to $\neg\langle\gamma, E\rangle\varphi$.
- complete axiomatizations.
- can we discard the (un-intuitive) monotonicity condition on the strategy forcing relations?

questions and comments

- adding game-strategy pair for the iteration operation: do we have to consider the same strategy for iteration game?
- exploring the identical game-strategy pairs - algebraic viewpoint.
- bisimulation-invariant formulas - a correct notion.
- representing evaluation games.

evaluation game

propositional evaluation game:

$\vee$ - move for ‘E’

$\wedge$ - move for ‘A’

$\neg$ - players change roles, winning conditions get interchanged.

a propositional formula - game form

a valuation - winning conditions

evaluation game

$\varphi = (p \vee q) \wedge \neg(p \wedge q)$, $V := p \rightarrow 1$, $q \rightarrow 0$.

equivalent c.n.f. = $(p \vee q) \wedge (\neg p \vee \neg q)$, only winning conditions get altered.

$$
\begin{aligned}
gs_V(\varphi) \quad &= \quad \langle(((p? \vee q?) \wedge ((p?)^d \vee (q?)^d), (\top \vee \bot) \wedge \\
&\qquad (\bot \vee \top)), E\rangle\top \\
&\leftrightarrow \quad (\langle(p?, \top), E\rangle\top \vee \langle(q?, \bot), E\rangle\top) \wedge \\
&\qquad (\langle(((p?)^d, \bot), E\rangle\top \vee \langle(((q?)^d, \top), E\rangle\top)
\end{aligned}
$$

evaluation game

formal representation:

atomic games: $p?$, atomic strategies: $\top$, $\bot$

composite games: $G \cup G'$, $G \cap G'$, G^d

composite strategies: $S \cup S'$, $S \cap S'$

game frame ($\mathcal{G}$): $\langle \{w\}, \rho_g^E = \rho_{g,\top}^E = \rho_{g,\top}^A = \{w, \{w\}\} \rangle$

Given a propositional formula φ, and a propositional valuation V, $V \models \varphi$ iff $(\mathcal{G}, V) \models gs_V(\varphi)$.

evaluation game

modal evaluation game:

propositional evaluation game

$+$

$\Diamond$ - move for ‘E’

$\Box$ - move for ‘A’

a modal formula $+$ Kripke frame - game form

a valuation - winning conditions

evaluation game

a finite model:

$\mathcal{M}$ = $(\{1,2\},\{(1,1),(1,2)\},V(p)=\{1,2\},V(q)=\emptyset)$

φ = $\Diamond\Diamond\top$.

$gs_{\mathcal{M},1}(\varphi)$ = $\langle(((C^E_{11};C^E_{11}\vee C^E_{12})\vee C^E_{12},(\top;\top\vee\top)\vee\bot),E\rangle\top$.

φ = $\Diamond p\vee\Diamond q$.

$gs_{\mathcal{M},1}(\varphi)$ =
$\langle(C^E_{11}\vee C^E_{12},\top\vee\top),E\rangle p\vee\langle(C^E_{11}\vee C^E_{12},\bot\vee\bot),E\rangle q$.

evaluation game

formal representation:

effort continues........

Thank you!