Specifying strategies in terms of their properties

Sunil Simon

Joint work with R. Ramanujam

The Institute of Mathematical Sciences, Chennai, India

Rational agents

Underlying assumption of equilibrium analysis - players interact in an ideal world.

- Players have perfect knowledge about all possible strategies.
- Players have unbounded computational resources.
- Common knowledge of rationality holds.

Many practical situations:

- Players are agents with limited computation resources.
- Players employ bounded memory strategies.

Structured strategies

Choices made by players depend on:

- Observations made during the play.
- Response to observed behaviour of other players.

Strategies are better viewed as relations constraining moves rather than complete functions.

Question: Can we come up with a framework where strategies are specified as structured objects built in some compositional fashion ?

Logical analysis of games

Modal logic: finite extensive form games.

- Encoding utilities, characteristic formula for backward induction procedure [Bonano].
- Characteristic formula for Nash equilibrium and sub-game perfect equilibrium [Harrenstein et al].
- Dynamic logic framework to describe strategies and to reason about outcomes [van Benthem].
- Dynamic logic framework describing games and strategies [Ghosh].

Logical analysis of games

Temporal logic: games on graphs.
Alternating time temporal logic [Alur et al].

Strategic reasoning in ATL:

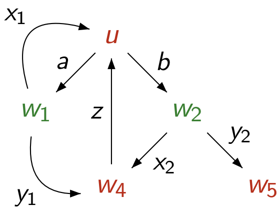
- In terms of epistemic conditions of players ([Jamroga, van der Hoek], [van der Hoek, Wooldridge]).
- Logic where (functional) strategies are explicitly part of the formalism ([van der Hoek et al], [Walther et al]).
- Ability to reason about specific actions of players ([Agotnes], [Borgo]).

Our attempt

- A syntactic framework where partially specified strategies are composed in a structured fashion.
- Explicate the strategic response of players.
- Independent of the exact depth of the game tree.

Games on graphs

Game model - directed graph where nodes are labelled with players.



Game arena

Games on graphs

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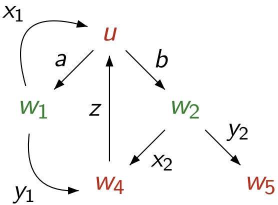
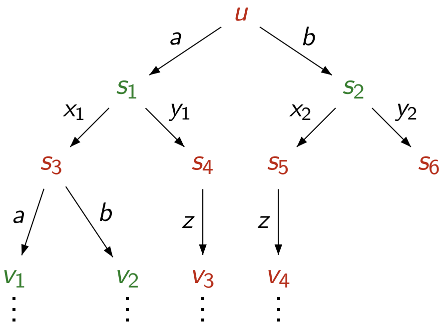


Game arena



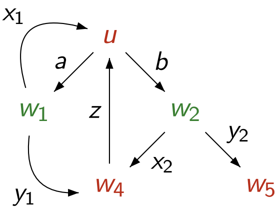
Extensive form game tree

P - countable set of observables

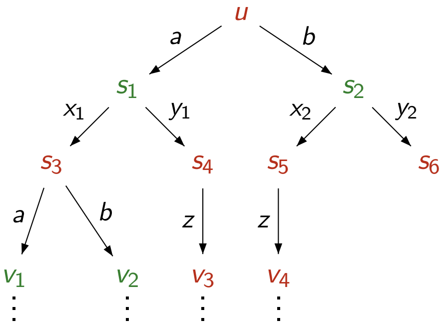
$V : Nodes \to 2^P$

Games on graphs

Strategies of players - subtrees of the game tree.



Game arena



Extensive form game tree

Games on graphs

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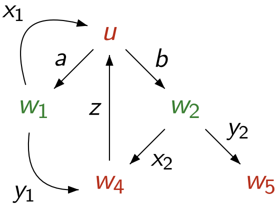
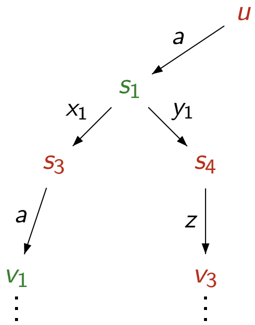


Game arena



A strategy of player 1

Strategy specification

$$Strat^i(P^i) := [\psi \mapsto a]^i$$

Interpretation

- $[\psi \mapsto a]^i$: If the observable ψ holds then choose action a (positional strategies).

Strategy specification

$$Strat^i(P^i) := [\psi \mapsto a]^i \mid \sigma_1 + \sigma_2 \mid \sigma_1 \cdot \sigma_2$$

Interpretation

- $[\psi \mapsto a]^i$: If the observable ψ holds then choose action a (positional strategies).
- $\sigma_1 + \sigma_2$: Disjunction.
- $\sigma_1 \cdot \sigma_2$: Conjunction.

Strategy specification

$$Strat^i(P^i) := [\psi \mapsto a]^i \mid \sigma_1 + \sigma_2 \mid \sigma_1 \cdot \sigma_2 \mid \pi \Rightarrow \sigma.$$

- π - specification of player $\bar{\imath}$.

Interpretation

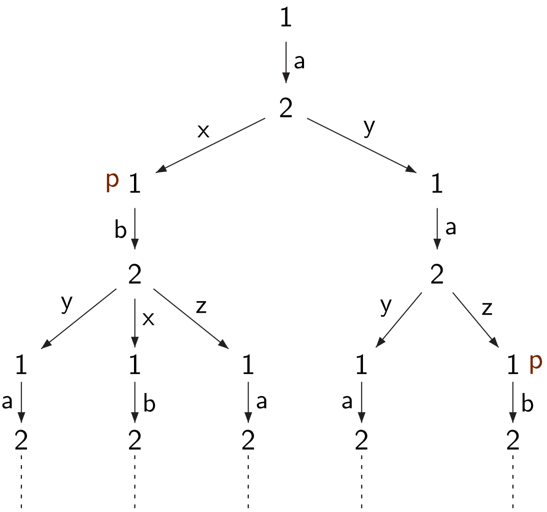
- $[\psi \mapsto a]^i$: If the observable ψ holds then choose action a (positional strategies).
- $\sigma_1 + \sigma_2$: Disjunction.
- $\sigma_1 \cdot \sigma_2$: Conjunction.
- $\pi \Rightarrow \sigma$: If in the history the observed behaviour of player $\bar{\imath}$ conforms to π then play according to σ.

Strategy specification

- Strategy specifications need not define complete strategies.
- Define when a (functional) strategy satisfies a specification.

Strategy conforming to a specification

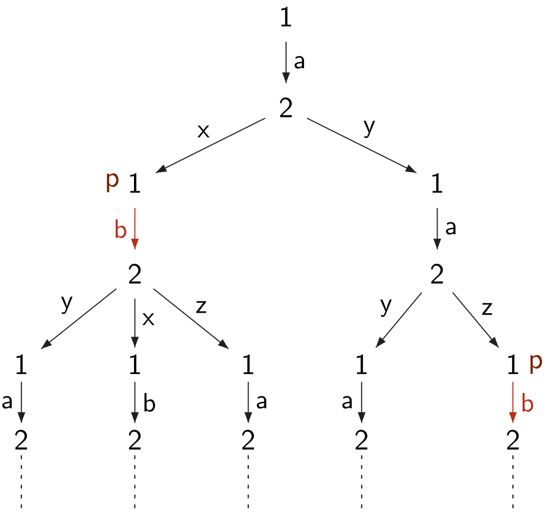
Player 1 strategy.



Strategy conforming to a specification

Player 1 strategy.

$[p \mapsto b]^1$



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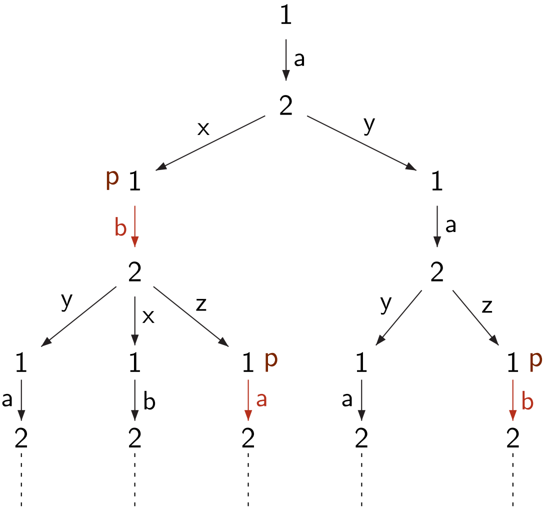
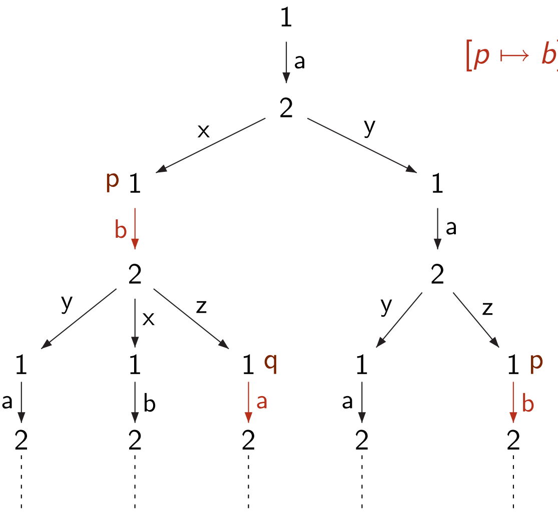


Strategy conforming to a specification

Player 1 strategy.

$\sigma_1 \cdot \sigma_2$

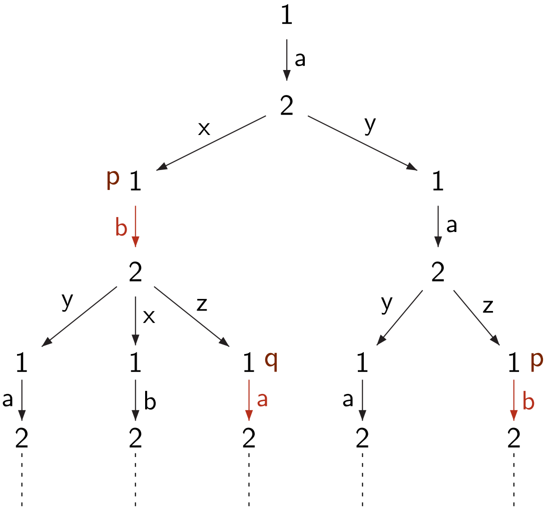
$[p \mapsto b]^1 \cdot [q \mapsto a]^1$



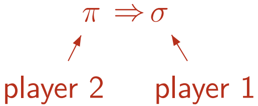
Strategy conforming to a specification

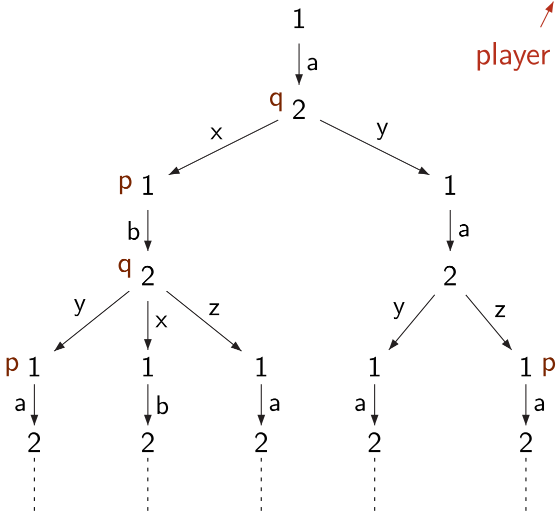
Player 1 strategy.

$\sigma_1 + \sigma_2$



Strategy conforming to a specification

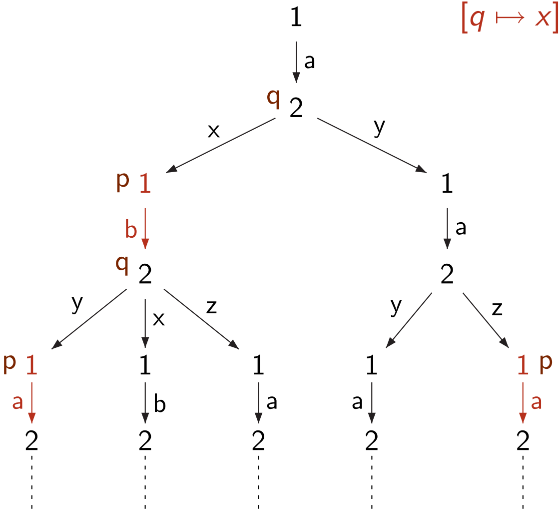




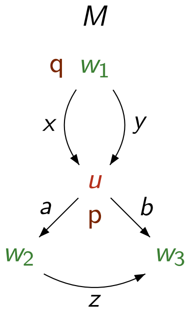
Strategy conforming to a specification

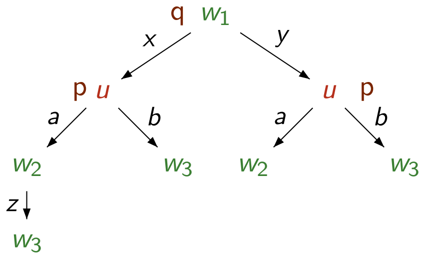
$$\pi \Rightarrow \sigma$$

$$[q \mapsto x]^2 \Rightarrow [p \mapsto b]^1$$

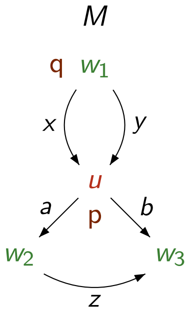


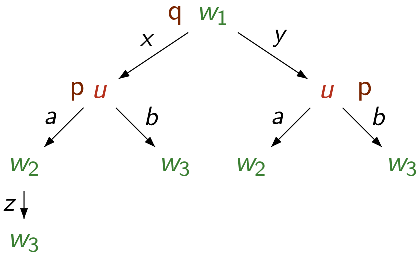
Examples - strategy specification

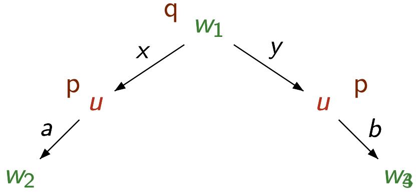




Examples - strategy specification







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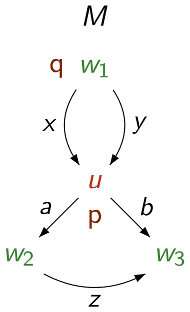
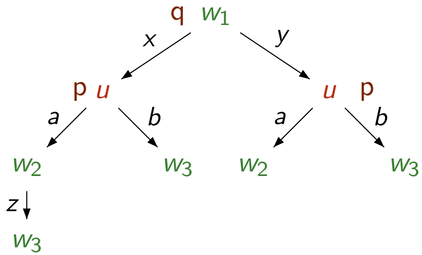
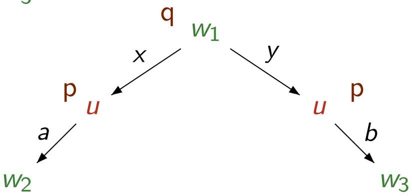




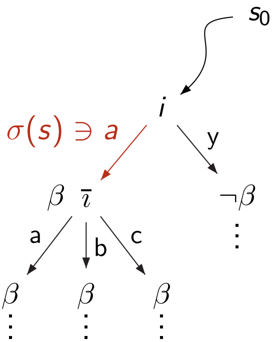


$$[q \mapsto x]^2 \Rightarrow [p \mapsto a]^1$$
$$[q \mapsto y]^2 \Rightarrow [p \mapsto b]^1$$

The logic

$$p \in P \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \langle a \rangle \alpha \mid \langle \overline{a} \rangle \alpha \mid \diamondminus\alpha \mid (\sigma)_i : c \mid \sigma \leadsto_i \beta$$

- $(\sigma)_i : c$ - The move c is enabled by the specification σ.
- $\sigma \leadsto_i \beta$ - The strategy specification σ ensures the outcome β.



The logic

Empty specification: $null^i$ - existence of strategies.

- $null^i \rightsquigarrow_i \beta$ - There exists a strategy to ensure the outcome β.
- $\sigma \rightsquigarrow_i \beta$ - The mechanism used by the player to ensure β is specified by the property σ.

Outcome based analysis

Finite extensive form games - special case in our setting.

- Utilities can be coded in terms of propositions.
- Characteristic formulas can be given for:
  - Best response.
  - Dominant strategies.
  - Equilibrium.

Games with compositional structure

Logical analysis of strategies: Explicates the strategic reasoning of players.

- The game representation is taken to be atomic.
- Logical formalism does not dictate the structure of the game.

Games with compositional structure

Logical analysis of strategies: Explicates the strategic reasoning of players.

- The game representation is taken to be atomic.
- Logical formalism does not dictate the structure of the game.

Question: What if the game is built in a compositional manner?

Game logic [Parikh]

A logic to reason about determined two person zero sum games.

Syntax

- $\Phi := p \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \langle\gamma\rangle\alpha$.
- $\Gamma := g \in \Gamma_0 \mid \gamma_1;\gamma_2 \mid \gamma_1 \cup \gamma_2 \mid \gamma^d \mid \gamma^*$.

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Interpretation for games

- Final outcomes which players can enforce.
- Set of states S.
- Effectivity relation - $E_g \subseteq S \times 2^S$
  - $(s, X) \in E_g$ iff starting at s, in game g, player 1 can enforce the outcome to be in X.

Game logic [Parikh]

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Model $M = (S, \{E_g \mid g \in \Gamma_0\}, V)$.

Neighbourhood semantics

- $M, s \models \langle\gamma\rangle\alpha$ iff $\exists(s, X) \in E_\gamma$ such that $X \subseteq \{s' \mid M, s' \models \alpha\}$.
- Player 1 has the ability in game γ to ensure α.
- Talks about players' abilities to achieve certain objectives.

Games with compositional structure

Players' strategic response need to take into account:

- Observable behaviour of the other players.
- Compositional structure of the game.

At the logical level:

- Game composition and structured strategies are *not* independent entities.
- Games and strategies need to be composed together.

The logic

Syntax

- $\Phi := p \in P \mid \neg\alpha \mid \alpha_1 \vee \alpha_2 \mid \langle\xi\rangle\alpha$.
- $\Gamma := (g, \sigma) \mid \xi_1; \xi_2 \mid \xi_1 \cup \xi_2 \mid \xi^*$.

Neighbourhood semantics

- $M, u \models \langle\xi\rangle\alpha$ iff there exists $(u, X) \in R_\xi$ such that $X \subseteq \{w \mid M, w \models \alpha\}$.

Neighbourhood semantics

To define the neighbourhood relation, we need to fix:

- Representation of game g.

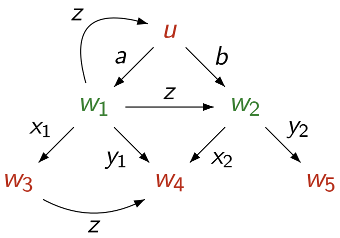
Atomic games: Extensive form games

- Finite tree - nodes represent game positions labelled with players.
- Edge relation - specifies the moves which are enabled at a particular position.

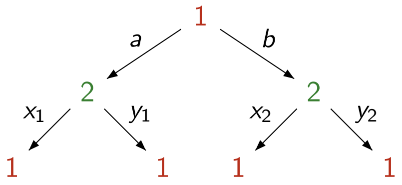
Model

- Model - Kripke structure.
  - A finite set of states W.
  - Labelled edge relation $\longrightarrow \subseteq W \times \Sigma \times W$.
  - Valuation function $V : W \rightarrow 2^P$.
  - Player labelling function $\lambda : W \rightarrow \{1, 2\}$.
- $(u, X) \in R_{(g,\sigma)}$ iff g is enabled at u and there exist a strategy μ conforming to σ such that the leaf nodes of μ is X.

Game - strategy pairs

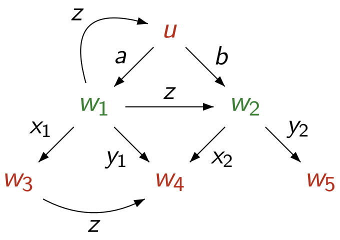


Model - M



Game - g

Game - strategy pairs



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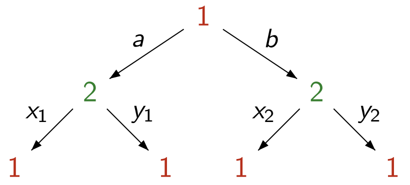
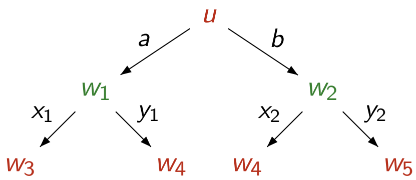


Game - g



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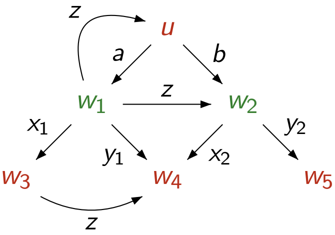
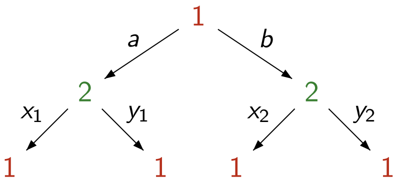
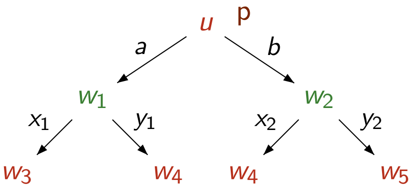


Model - M

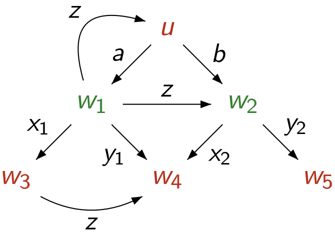


Game - g

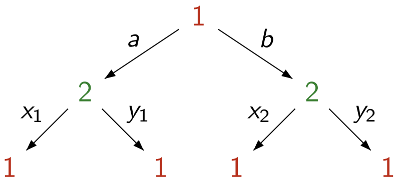
$$\sigma = [p \mapsto a]^1$$



Game - strategy pairs



Model - M



Game - g

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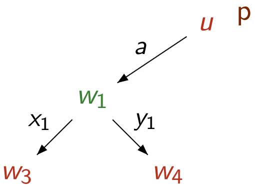
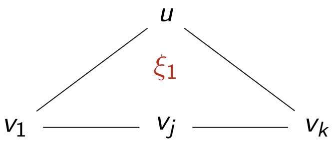


$$(u, \{w_3, w_4\}) \in R_{(g,\sigma)}$$

Semantics of game - strategy pairs

$\Gamma := (g, \sigma) \mid \xi_1; \xi_2 \mid \xi_1 \cup \xi_2 \mid \xi^*$

- $(u, X) \in R_{\xi_1;\xi_2}$



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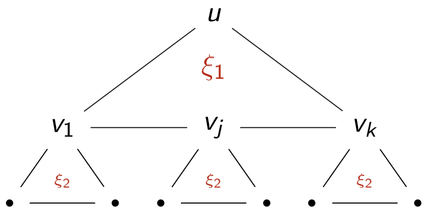

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- $(u, X) \in R_{\xi_1;\xi_2}$ iff
  - $\exists Y = \{v_1, \ldots, v_k\}$ such that $(u, Y) \in R_{\xi_1}$.
  - $\forall v_j \in Y$, $\exists X_j \in X$ such that $(v_j, X_j) \in R_{x_2}$.
  - $X = \bigcup_{j=1,\ldots,k} X_j$.

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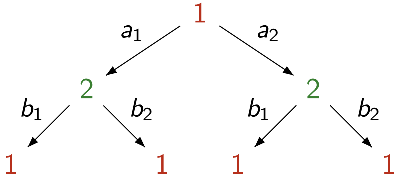
  - $X = \bigcup_{j=1,\ldots,k} X_j$.
- $R_{\xi_1 \cup \xi_2} = R_{\xi_1} \cup R_{\xi_2}$.
- $R_{\xi^*} = \bigcup_{n \geq 0} (R_\xi)^n$.

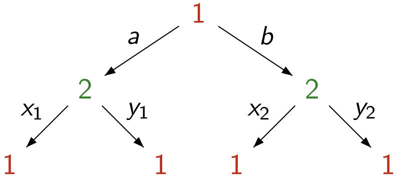
Examples - game composition

- Consider a two stage game g_1 followed by g_2.
- Player 1's planning at the end of g_1 depends not only on how g_2 is structured but also on how player 2 played in g_1.
- $(g_1, \pi); (g_2, \sigma)$.
  - π strategy specification of player 2.
  - σ strategy specification of player 1.
- (g_2, σ) is the response of player 1 to (g_1, π).

Examples - game composition



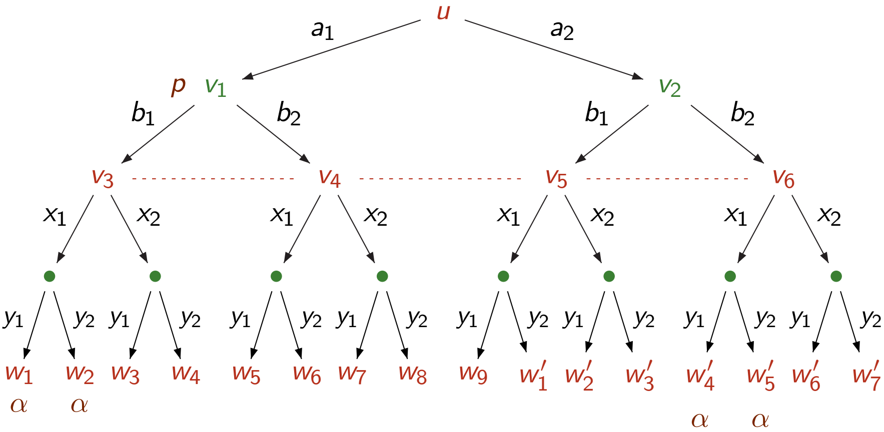
g_1



g_2

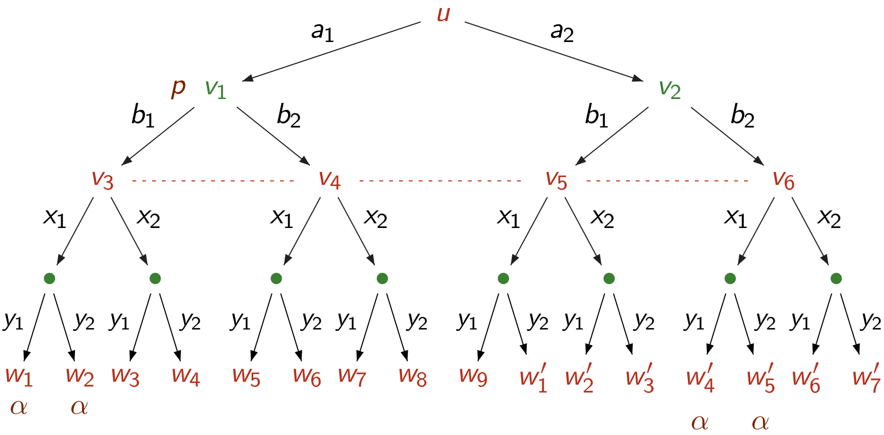
$g_1; g_2$

Examples - game composition



- Player 1 does not have a strategy in g to ensure α.

Examples - game composition



- Let $\pi \equiv [p \mapsto b_1]^2 \cdot [\neg p \mapsto b_2]^2$ and $\sigma \equiv [True \mapsto x_1]^1$.
- $M, u \models \langle (g_1, \pi); (g_2, \sigma) \rangle \alpha$

Examples

- $\langle(g,\sigma)^*\rangle\langle g', \mathit{null}^2\rangle\alpha$.
    - σ - strategy specification of player 1.
    - null^2 - player 2 is allowed to pick any strategy.

By iterating the structure (g,σ) player 1 can ensure a state where g' is enabled and irrespective of what player 2 does, α is guaranteed.

Axiom system

- $\langle g, \sigma \rangle \alpha \equiv ?$

(Informally): Game g is enabled and there exists a strategy μ conforming to σ such that *frontier*(μ) satisfies α.

Axiom system

- $\langle g, \sigma \rangle \alpha \equiv g^{\checkmark} \wedge push(g, \sigma, \alpha)$.

(Informally): Game g is enabled and there exists a strategy μ conforming to σ such that $frontier(\mu)$ satisfies α.

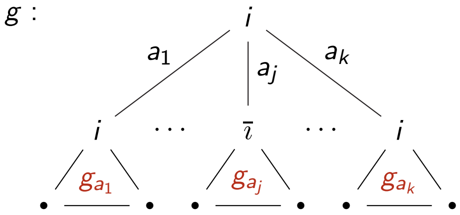
- $\langle a \rangle \alpha$ - can be encoded in the logic.
- $g^{\checkmark}$ can be defined.

Definition of *push*

g is a single node:

- $push(g, \sigma, \alpha) = \alpha$.

Axiom system

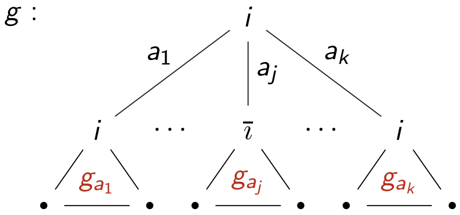


$\sigma = [p \mapsto a]^i$

$push(g, \sigma, \alpha)$ holds at state u:

- if p holds at state u
  - then $\exists w$ such that $u \xrightarrow{a} w$ and $\langle g_a, \sigma \rangle \alpha$ holds at w.

Axiom system



$\sigma = [p \mapsto a]^i$

$push(g, \sigma, \alpha)$ holds at state u:

- if p holds at state u
  - then $\exists w$ such that $u \xrightarrow{a} w$ and $\langle g_a, \sigma \rangle \alpha$ holds at w.
- if p does not hold at state u
  - then $\exists a_j$ such that $u \xrightarrow{a_j} w$ and $\langle g_a, \sigma \rangle \alpha$ holds at w.

Axiom system

General idea behind *push*

- If the root is a player i-node and we have a player i-specification then
  - an edge which conforms to the specification is chosen and the requirement is "pushed" to the relevant subtree.
- If the root is an $\overline{\imath}$-node and we have a player i-specification then
  - all outgoing edges need to be taken into account and the requirement is "pushed" to all the resulting subtrees.

Axiom system

- Propositional axioms:
  - All the substitutional instances of tautologies of PC.
  - $turn_i \equiv \neg turn_{\bar{\imath}}$.
- Axiom for single edge games:
  - $\langle a \rangle (\alpha_1 \vee \alpha_2) \equiv \langle a \rangle \alpha_1 \vee \langle a \rangle \alpha_2$.
  - $\langle a \rangle turn_i \supset [a] turn_i$.
- Dynamic logic axioms:
  - $\langle \xi_1 \cup \xi_2 \rangle \alpha \equiv \langle \xi_1 \rangle \alpha \vee \langle \xi_2 \rangle \alpha$.
  - $\langle \xi_1 ; \xi_2 \rangle \alpha \equiv \langle \xi_1 \rangle \langle \xi_2 \rangle \alpha$.
  - $\langle \xi^* \rangle \alpha \equiv \alpha \vee \langle \xi \rangle \langle \xi^* \rangle \alpha$.

Inference rules

$$(MP)\ \frac{\alpha, \quad \alpha \supset \beta}{\beta} \qquad (NG)\ \frac{\alpha}{[a]\alpha}$$

$$(IND)\ \frac{\langle \xi \rangle \alpha \supset \alpha}{\langle \xi^* \rangle \alpha \supset \alpha}$$

Decidability

A formula is satisfiable iff it is satisfiable in an exponential sized model.

Given α to decide if α is satisfiable:

- Guess an exponential sized model M.
- Explicitly build the relation $R_\xi \subseteq S \times 2^W$.
  - Time: exponential in the size of the model.
- Check whether M satisfies α.

Conclusion

- Mixed strategies
  - Switching of strategies by players.
  - Incorporate the notion of expectations of players.
- Relation between composition of games and that of sub-games.
- Adapting strategy specifications to deal with games of imperfect information.