Modal Structures in Groups and Vector Spaces

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Modal structures in groups and vector spaces

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Abstract

Vector spaces contain a number of general structures that invite analysis in modal languages. The resulting logical systems provide an interesting counterpart to the much better-studied modal logics of topological spaces. In this programmatic paper, we investigate issues of definability and axiomatization using standard techniques for modal and hybrid languages. The analysis proceeds in stages. We first present a modal analysis of commutative groups that establishes our main techniques, next we introduce a new modal logic of linear dependence and independence in vector spaces and, finally, we study a modal logic for describing full-fledged vector spaces. While still far from covering every basic aspect of linear algebra, our discussion identifies several leads for more systematic research.

1 Introduction

Vector spaces and techniques from linear algebra are ubiquitous in applied mathematics and physics, but they also occur in areas such as cognitive science [27], machine learning [16], computational linguistics [31], the social sciences [6, 46] and formal philosophy [44]. There is also a body of logical work on vector spaces, in the first-order model-theoretic tradition [39, 51], in relevant logic [54] and in modal logics of space [12]. This paper offers a further exploration from the perspective of modal logic, in the broad spirit of [35, 36]. As it happens, such connections between logic and mathematics can be pursued in two directions, both present in the cited literature. Vector spaces have been applied as a source of new semantic models for existing independently motivated logical languages and axiom systems. But one can also put the focus on vector spaces themselves, asking which notions from linear algebra can be captured in which specially designed new logics. The latter approach will be our main interest in what follows, though the two directions are of course not incompatible.

Vector spaces

Definition. A vector space over a field F is a set V with operations $+ : V^2 \to V$ of vector addition and a scalar multiplication $\cdot : F \times V \to V$ such that for each $u, v, w \in V$ and $a, b, c \in F$:

1. $u + (v + w) = (u + v) + w$
2. $u + v = v + u$
3. there exists a 'zero vector' $0 \in V$ such that $v + 0 = v$ for all $v \in V$
4. for every $v \in V$, there is an 'additive inverse' $-v \in V$ such that $v + (-v) = 0$
5. $a \cdot (b \cdot v) = (a\, b) \cdot v$
6. there exists a 'multiplicative unit' $1 \in F$ such that $1 \cdot v = v$ for all $v \in V$
7. $a \cdot (u + v) = a \cdot u + a \cdot v$
8. $(a + b) \cdot v = a \cdot v + b \cdot v$

Groups

Definition. A group is a structure $(G, +, -, 0)$ where $+$ is a binary operation on G and $-$ is a unary operation on G and $0 \in G$ is a constant (0-ary operation) such that for all $a, b, c \in G$:

1. $a + (b + c) = (a + b) + c$;
2. $a + 0 = 0 + a = a$;
3. $a + (-a) = (-a) + a = 0$.

A group G is commutative if for each $a, b \in G$: $a + b = b + a$.

Syntax

Let Prop be a set of propositional variables, Nom a set of nominals and 0 a hybrid constant.

The modal group language MGL is generated by the following grammar:

$$\varphi := \bot \mid p \mid \mathsf{i} \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{E}\varphi \mid \langle - \rangle\varphi \mid \langle + \rangle(\varphi, \varphi) \mid 0,$$

where $p \in$ Prop, $\mathsf{i} \in$ Nom.

We will use the shorthands

$\varphi \oplus \psi := \langle + \rangle(\varphi, \psi)$,

$[+](\varphi_1, \varphi_2) := \neg\langle + \rangle(\neg\varphi_1, \neg\varphi_2)$,

$\mathsf{U}\varphi := \neg\mathsf{E}\neg\varphi$.

Group models

Definition. A group model is a tuple $(S, +, V)$, with a commutative group $(S, +, -, 0)$ and a valuation map

$V : \mathsf{Prop} \cup \mathsf{Nom} \to \mathcal{P}(S)$ s.t. $|V(\mathsf{i})| = 1$ for all nominals i and $V(\mathsf{0}) = \{0\}$ is the unit of the group.

Formulas φ are interpreted as follows:

Semantics

Definition. Let $\mathfrak{M} = (S, +, V)$ be a group model, $s \in S$:

$\mathfrak{M}, s \models \bot$ never

$\mathfrak{M}, s \models \mathsf{i}$ iff $s \in V(\mathsf{i})$

$\mathfrak{M}, s \models p$ iff $s \in V(p)$

$\mathfrak{M}, s \models \varphi \vee \psi$ iff $\mathfrak{M}, s \models \varphi$ or $\mathfrak{M}, s \models \psi$

$\mathfrak{M}, s \models \neg\varphi$ iff not $\mathfrak{M}, s \models \varphi$

$\mathfrak{M}, s \models \mathsf{E}\varphi$ iff $\mathfrak{M}, t \models \varphi$, for some $t \in G$

$\mathfrak{M}, s \models \mathsf{0}$ iff $s \in V(\mathsf{0})$

$\mathfrak{M}, s \models \varphi \oplus \psi$ iff $\exists s_1, s_2$ s.t. $\mathfrak{M}, s_1 \models \varphi$, $\mathfrak{M}, s_2 \models \psi$ and $s = s_1 + s_2$

$\mathfrak{M}, s \models \langle - \rangle\varphi$ iff $\mathfrak{M}, -s \models \varphi$

Definable sets

We say that a subset of a group model $\mathfrak{M}$ is definable if it is the denotation of some formula.

One can think of these subsets as ‘patterns’ in space.

Let us now look at some examples.

Definable sets: Examples

Example 1. Consider the group model based on the set of integers $(\mathbb{Z}, +, -, 0)$ with $V(p) = \{1\}$, and no nominals interpreted.

Each set $\{z\}$ for a positive integer z is definable, by 'summing' p z times.

Also all singletons $\{-z\}$ are definable. Using disjunctions and negation, every finite and every cofinite subset of $\mathbb{Z}$ is definable.

This collection is closed under the operations $0, -, +$ defined above plus the Boolean operations.

E.g., all sums of numbers from two cofinite sets is itself cofinite, and so on.

A similar analysis works for $(\mathbb{Z} \times \mathbb{Z}, +, -, (0, 0))$ with the valuation $V(p) = \{(0, 1)\}, V(q) = \{(1, 0)\}$.

Definable sets: Examples

Examples 2. Consider $(\mathbb{Z} \times \mathbb{Z}, +, -, (0,0))$ and let $V(p) = \{(0,1),(1,0)\}$.

Can we define, say, the single point $(1,1)$?

We first list some further sets that are definable from $V(p)$ using the Booleans and modalities in our language:

$\langle - \rangle p$ defines $\{(0,-1),(-1,0)\}$,
$p \oplus p \; : \; \{(0,2),(1,1),(2,0)\}$,
$p \oplus \langle - \rangle p \; : \; \{(0,0),(-1,1),(1,-1)\}$,
$p \oplus (p \oplus \langle - \rangle p) \; : \; \{(-1,2),(0,1),(1,0),(2,-1)\}$,
$(p \oplus (p \oplus \langle - \rangle p)) \wedge \neg p \; : \; \{(-1,2),(2,-1)\}$ Call this last formula ψ.

Now the single point $(1,1)$ is definable by the formula $(\psi \oplus \psi) \wedge (p \oplus p)$.

Definable sets: Examples

Question: Which sets can we define with $V(p)$?

A subset U is reflective if they are closed under the map sending points (x,y) to their reflections (y,x) along the diagonal $x=y$.

Answer: We obtain finite and cofinite reflective subsets of $\mathbb{Z}\times\mathbb{Z}$.

Spoiler: This result is not in the paper.

Minkowski operations

The Minkowski operations on subsets of groups are addition

$$A \oplus B = \{x + y \mid x \in A \text{ and } y \in B\}$$

and difference

$$A \ominus B = \{x \mid \text{for all } y \in B, x + y \in A\}$$

Then we have

$$A \ominus B = \neg(\langle - \rangle B \oplus \neg A))$$

Note that

$$A \neq A \oplus A$$

Minkowski operations

This also gives us semantics of substructural logic, e.g., commutative Lambek calculus where

$$A \oplus C \subseteq B \text{ iff } C \subseteq A \Rightarrow B$$

We have

$$A \Rightarrow B := B \ominus A$$

Problem: What is the substructural logic of comutative groups?

The formula

$$(\psi \Rightarrow (\varphi \oplus \alpha)) \Leftrightarrow ((\psi \Rightarrow \varphi) \oplus \alpha)$$

does not hold on commutative groups, but

$$(\psi \Rightarrow (\varphi \oplus \mathsf{n})) \Leftrightarrow ((\psi \Rightarrow \varphi) \oplus \mathsf{n})$$

holds for a nominal n.

Some valid modal principles in group models

Fact. $\mathsf{E}\varphi \leftrightarrow \varphi \oplus \top$ is valid in group models.

Proof.

First let $s \models \varphi \oplus \top$. Then there are points v, u s.t. $w = v + u$ and $v \models \varphi$. In particular, there is a point where φ is true, and so $\mathsf{E}\varphi$ is true at s.

Conversely, let $s \models \mathsf{E}\varphi$. Then φ is true at some point t, and we can use the fact that in any group: $s = t + (-t + s)$ to see that $s \models \varphi \oplus \top$. □

We will keep the global modalities in our language as they make principles easier to grasp.

Bisimulations can also be defined in this framework (see the paper).

Frame definability and p-morphisms

Definition. A map f is a p-morphism from G_1 to G_2 if it sends points in G_1 to points in G_2 subject to the following conditions:

1. f maps the zero element of G_1 to that of G_2,
2. $f(v + v') = f(v) + f(v')$ and $f(-v) = -f(v)$,
3. If $f(v) = v'_1 + v'_2$ then there are v_1, v_2 with $v = v_1 + v_2$, $f(v_1) = v'_1$ and $f(v_2) = v'_2$.

Frame definability and p-morphisms

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3. If $f(v) = v'_1 + v'_2$ then there are v_1, v_2 with $v = v_1 + v_2$, $f(v_1) = v'_1$ and $f(v_2) = v'_2$.

Fact. On group models, given Con. (2), Con. (3) for a p-morphism f as defined above is equivalent to surjectivity of the function f.

Relational models

Definition. A relational model is a tuple $(S, R, I, 0, V)$, of a non-empty set of points S, a ternary relation $Rstu$, a unary function I and a distinguished object 0.

This triple is called a frame, plus a valuation map $V : \mathsf{Prop} \cup \mathsf{Nom} \to \mathcal{P}(S)$ s.t. $|V(\mathsf{i})| = 1$ and $V(0) = \{0\}$.

The truth definition for the two modalities in our language:

$$\mathfrak{M}, s \models \varphi \oplus \psi \quad \text{iff} \quad \exists s_1, s_2 \text{ s.t. } Rss_1s_2 \text{ and } \mathfrak{M}, s_1 \models \varphi, \mathfrak{M}, s_2 \models \psi$$

$$\mathfrak{M}, s \models \langle - \rangle \varphi \quad \text{iff} \quad \mathfrak{M}, I(s) \models \varphi$$

Definability and p-morphisms

A map f is a p-morphism from a general relational frame $\mathsf{F}_1 = (S_1, R_1, I_1, 0_1)$ to a general relational frame $\mathsf{F}_2 = (S_2, R_2, I_2, 0_2)$ if it sends points in F_1 to points in F_2 subject to the following conditions:

1. f maps the zero element of F_1 to that of F_2,
2. $R_1(v_1, v_2, v_3)$ implies $R_2(f(v_1), f(v_2), f(v_3))$ and $f(I_1(v)) = I_2(f(v))$,
3. If $R_2(f(v), v_1', v_2')$ then there are v_1, v_2 with $R(v, v_1, v_2)$ s.t. $f(v_1) = v_1'$ and $f(v_2) = v_2'$.

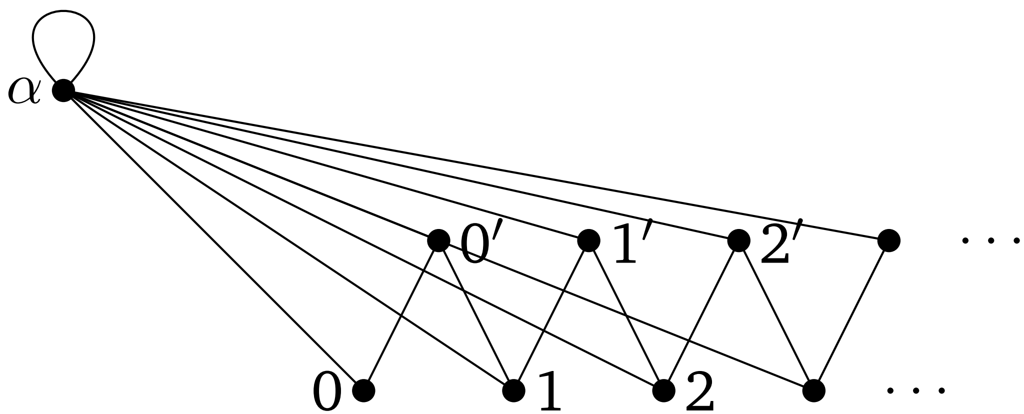
Frame definability and p-morphisms

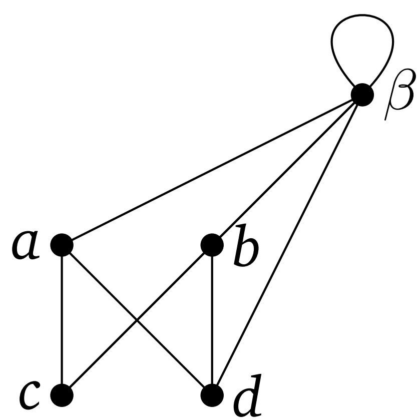
Fact. If F_1 and F_2 are relational or group models and f is a p-morphism from frame F_1 to F_2 and $\mathsf{F}_1 \models \varphi$, then $\mathsf{F}_2 \models \varphi$.

Fact. Functionality of $+$ is not expressible in relational frames in our modal language of groups without nominals.

Frame definability and p-morphisms

Idea:





Definability and p-morphisms

Fact. Functionality of the binary operation $+$ is defined on relational frames by the modal formulas

$$((\mathsf{E}\mathsf{n} \wedge \mathsf{E}\mathsf{m}) \to \mathsf{E}(\mathsf{m} \oplus \mathsf{n})),$$

$$(\mathsf{E}(\mathsf{n} \wedge \mathsf{m} \oplus \mathsf{k}) \to \mathsf{U}(\mathsf{m} \oplus \mathsf{k} \to \mathsf{n})).$$

This is one more motivation for working with a hybrid language.

Modal logic of commutative groups

Axiom system and completeness

Definition. The Modal Logic of Commutative Groups LCG has the following axioms:

- All tautologies of classical propositional logic
- $(\varphi \vee \psi) \oplus \chi \leftrightarrow (\varphi \oplus \chi) \vee (\psi \oplus \chi)$
- $\langle - \rangle(\varphi \vee \psi) \leftrightarrow \langle - \rangle\varphi \vee \langle - \rangle\psi$
- $\varphi \rightarrow \mathsf{E}\varphi$
- $\mathsf{EE}\varphi \rightarrow \mathsf{E}\varphi$
- $\varphi \rightarrow \mathsf{UE}\varphi$
- $\varphi \oplus \psi \rightarrow \mathsf{E}\varphi$
- $\langle - \rangle\varphi \rightarrow \mathsf{E}\varphi$
- $E\mathsf{i}$ where i ranges over all nominals in our language
- $\mathsf{E}(\varphi \wedge \mathsf{i}) \rightarrow \mathsf{U}(\mathsf{i} \rightarrow \varphi)$

Axiom system and completeness

- $\langle - \rangle \neg \varphi \leftrightarrow \neg \langle - \rangle \varphi$
- $(\varphi \wedge \mathsf{E}\psi) \rightarrow \mathsf{E}(\varphi \oplus \psi)$
- $(\varphi \wedge \mathsf{m} \oplus \mathsf{n}) \rightarrow \mathsf{U}(\mathsf{m} \oplus \mathsf{n} \rightarrow \varphi)$

- $\varphi \oplus (\psi \oplus \chi) \leftrightarrow (\varphi \oplus \psi) \oplus \chi$
- $\varphi \oplus \psi \leftrightarrow \psi \oplus \varphi$
- $\varphi \oplus \mathsf{0} \leftrightarrow \varphi$
- $\mathsf{i} \oplus \langle - \rangle \mathsf{i} \leftrightarrow \mathsf{0}$

The rules of inference for LCG are

Modus Ponens, Replacement of Provable Equivalents, plus this Substitution Rule:

> arbitrary formulas can be substituted for formula variables φ, while for nominals, we can only substitute nominal terms formed from nominals using the functional modalities $\langle - \rangle$, $\oplus$.

Axiom system and completeness

Moreover, we have the following rules governing (and entangling) nominals and modalities:

Necessitation Rules: $\frac{\neg\varphi}{\neg(\varphi \oplus \psi)}$, $\frac{\neg\varphi}{\neg\langle-\rangle\varphi}$, $\frac{\neg\varphi}{\neg\mathsf{E}\varphi}$

Naming Rule: $\frac{\mathsf{j}\rightarrow\theta}{\theta}$ and

Witness Rules: $\frac{\mathsf{E}(\mathsf{j}\wedge\varphi)\rightarrow\theta}{\mathsf{E}\varphi\rightarrow\theta}$, $\frac{\langle-\rangle(\mathsf{j}\wedge\varphi)\rightarrow\theta}{\langle-\rangle\varphi\rightarrow\theta}$, $\frac{(\mathsf{j}\oplus\mathsf{k}\wedge\mathsf{E}(\mathsf{j}\wedge\varphi)\wedge\mathsf{E}(\mathsf{k}\wedge\psi))\rightarrow\theta}{(\varphi\oplus\psi)\rightarrow\theta}$.

In the Naming and Witness rules, j, k are nominals that do not occur in any of the formulas φ, ψ or θ.

Provable formulas

- $\mathsf{LCG} \vdash \top \oplus \top$
- $\mathsf{LCG} \vdash \mathsf{E}\varphi \leftrightarrow \varphi \oplus \top$
- $\mathsf{LCG} \vdash \mathsf{n} \leftrightarrow \langle - \rangle \langle - \rangle \mathsf{n}$
- $\mathsf{LCG} \vdash \varphi \leftrightarrow \langle - \rangle \langle - \rangle \varphi$

For the derivations see the paper.

Soundness and completeness

Theorem. LCG is sound and complete with respect to commutative groups.

Soundness is a straightforward verification.

Completeness follows the standard canonical model proof technique of hybrid logic.

Proof of completeness

We sketch the main steps of the proof.

It follows the standard proof technique in hybrid logic.

We want to construct a consistent set Σ to a maximal consistent one and prove a version of a truth lemma.

This proof is also similar the standard Henkin-style completeness proof for first-order logic

Proof of completeness

Using the Witness rules we find a family of maximally consistent sets for an extended language L (with extra nominals *a la* Henkin constants) where each maximally consistent set is named:

It contains at least one nominal denoting it uniquely, and these sets are also witnessing in the following sense:

- If $\mathsf{E}(\mathsf{n} \wedge \mathsf{E}\varphi) \in \Gamma$, then, for some nominal i not occurring in $\mathsf{n} \wedge \varphi$: $\mathsf{En} \wedge \mathsf{E}(\mathsf{i} \wedge \varphi) \in \Gamma$
- If $\mathsf{E}(\mathsf{n} \wedge \langle - \rangle \varphi) \in \Gamma$ then, for some some nominal i: $\mathsf{E}(\mathsf{n} \wedge \langle - \rangle \mathsf{i}) \wedge \mathsf{E}(\mathsf{i} \wedge \varphi) \in \Gamma$
- If $\mathsf{E}(\mathsf{n} \wedge \varphi \oplus \psi) \in \Gamma$, then for some nominals i, j: $\mathsf{E}(\mathsf{n} \wedge \mathsf{i} \oplus \mathsf{j}) \wedge \mathsf{E}(\mathsf{i} \wedge \varphi) \wedge \mathsf{E}(\mathsf{j} \wedge \psi) \in \Gamma$

Proof of completeness

Let $\Sigma R_{\mathsf{E}} \Delta$ if for every formula $\alpha \in \Delta : \mathsf{E}\alpha \in \Sigma$.

Now we fix one maximal consistent set $\Sigma^{\bullet}$ containing Σ. We will refer to $\Sigma^{\bullet}$ as the guidebook.

Let Γ_i be the set called after nominal i. We put

$$R_{\langle - \rangle}(\Gamma_{\mathsf{i}}, \Gamma_{\mathsf{j}}) \text{ if } \mathsf{E}(\mathsf{i} \wedge \langle - \rangle \mathsf{j}) \in \Sigma^{\bullet}$$

$$R_{\oplus}(\Gamma_{\mathsf{n}}, \Gamma_{\mathsf{i}}, \Gamma_{\mathsf{j}}) \text{ if } \mathsf{E}(\mathsf{n} \wedge (\mathsf{i} \oplus \mathsf{j})) \in \Sigma^{\bullet}$$

Proof of completeness

Take the model $\mathfrak{M}$ based on the R_{E}-equivalence class of $\Sigma^{\bullet}$.

Then our axioms ensure that $\mathfrak{M}$ is a group and that the truth lemma holds:

The Truth Lemma.

$$\mathfrak{M}, \Gamma_{\mathsf{i}} \models \varphi \text{ iff } \mathsf{E}(\mathsf{i} \wedge \varphi) \in \Sigma^{\bullet}.$$

This essentially finishes the proof.

Group inverses

We expressed the basic law $v + (-v) = 0$ in our logic using nominals:

$$\mathsf{i} \oplus \langle - \rangle \mathsf{i} \leftrightarrow \mathsf{0}.$$

This cannot be lifted to sets in a direct manner, since

$$\varphi \oplus \langle - \rangle \varphi \leftrightarrow \mathsf{0}$$

is not a valid formula if the value of φ is not a singleton set,

However, there is a formula in our language that does the job without nominals (but with 0 for the zero element):

$$\mathsf{E}\varphi \to \mathsf{E}((\varphi \oplus \langle - \rangle \varphi) \wedge \mathsf{0})$$

is easily seen to enforce the basic law of inverses.

Complex algebra

Background of the difficulties: complex algebra.

With the modalities, we are really investigating a set lifting of the basic algebra of group addition.

This fits the paradigm of complex algebras.

Given a group $(G, +, -, 0)$.

Define the complex group $(\mathcal{P}(G), \oplus, -, \{0\})$, where

$A \oplus B = \{a + b : a \in A, b \in B\}$,
$-A = \{-a : a \in A\}$.

Gautam's curse

Gautam's Theorem shows that only equations when each variable occurs at most once on each side of the equation is preserved by complex algebras.

N. Gautam, The validity of equations of complex algebras. Archiv für mathematische Logik und Grundlagenforschung, 3, 117-124, 1957.

Note that not all our axioms fit this setting, but we also use Boolean connectives as well as nominals and avoid the curse.

Complex algebra results and questions

For a Boolean algebra $(B, \wedge, \vee, \neg, 0)$

Consider the complex algebra $(\mathcal{P}(B), \wedge, \vee, \neg, \{0\})$, where

$$A \wedge B = \{a \wedge b : a \in A, b \in B\},$$
$$A \vee B = \{a \vee b : a \in A, b \in B\},$$
$$\neg A = \{\neg a : a \in A\}.$$

Goranko & Vakarelov (1999) axiomatized via non-standard rules set-lifted Boolean algebra over families of sets, both set-lifted 'inner Booleans' and standard Booleans.

This approach does not settle decidability of the logic.

Long-standing open problem even for complex Boolean Algebra.

Open problems

- Could completeness be proved without nominals?
- Could we unravel the canonical model (in the language without nominals) into a group?
- Can we prove completeness for particular (classes of) groups? How about $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, etc.?

Other important modal operators

There are even more interesting modal operators in groups.

Definition. $\mathfrak{M}, s \models \mathsf{C}\varphi$ iff s is obtained from the set $\{t \mid \mathfrak{M}, t \models \varphi\}$ by finitely many uses of the binary operation $+$, the unary operation $-$ and the nullary operation 0.

Note that $\mathsf{C}\varphi$ is not an ordinary modality, since it does not distribute over conjunctions or disjunctions, as is clear from the behavior of closure in groups.

Our analysis will use ideas from neighborhood semantics for modal logic.

More on this in Johan's part.

Modal logic of vector spaces

Modal logic of vector spaces

We now turn to vector spaces.

Definition. The *terms* of the modal language of vector spaces MVL are given by the following schema, starting with some set of variables x, while 0, 1 are individual constants:

$$t \ := \ x \mid 0 \mid 1 \mid t + t \mid \ -t \mid t.t \mid t^{-1}$$

This definition also includes the term 0^{-1} or $(x + -x)^{-1}$ which do not denote objects in fields, and as a result, our semantics must deal with terms lacking a denotation.

Definition. Formulas are defined as follows, where $p \in \mathsf{Prop}$ and nominals $\mathsf{i} \in \mathsf{Nom}$ and t is arbitrary term:

$$\varphi \ := \ p \mid \mathsf{i} \mid \neg\varphi \mid \varphi \wedge \varphi \mid \mathsf{E}\varphi \mid \mathsf{0} \mid \langle - \rangle\varphi \mid \varphi \oplus \varphi \mid \langle t \rangle\varphi$$

Vector space semantcis

A vector model over a field is a structure $\mathfrak{M} = (S, F, V, h)$ with S a commutative group, F a field, and V a valuation for proposition letters on S.

Next, the assignment map h sends basic variable terms to objects in the field F. This map extends uniquely to a partial map from the whole set of terms to objects in F, also denoted by h.

Here the convention is that (a) complex terms with undefined components do not get a value, (b) if a term t has value 0, then t^{-1} does not get a value.

As an illustration, assignments h are undefined on the terms 0^{-1}, $(x + (-x))^{-1}$.

Vector space semantics

The modality $\langle t \rangle \varphi$ is interpreted as follows:

$$(S, V, h), v \models \langle t \rangle \varphi \quad \text{iff} \quad \text{there exists a vector } w \text{ such that}$$
$$\text{(i) } (S, V, h), w \models \varphi, \text{ (ii) } h(t) \bullet w \text{ is defined and } h(t) \bullet w = v$$

Dynamic vector logic

Definition. The proof calculus of Dynamic Vector Logic DVL consists of

- All axioms and rules of the proof system LCG,

  (a) Axioms for definedness of terms:

  (a1) $\mathsf{E}\langle s+t\rangle\top \leftrightarrow \mathsf{E}\langle s\rangle\top \wedge \mathsf{E}\langle t\rangle\top$

  (a2) $\mathsf{E}\langle s\cdot t\rangle\top \leftrightarrow \mathsf{E}\langle s\rangle\top \wedge \mathsf{E}\langle t\rangle\top$

  (a3) $\mathsf{E}\langle -s\rangle\top \leftrightarrow \mathsf{E}\langle s\rangle\top$

  (a4) $\mathsf{E}\langle s^{-1}\rangle\top \leftrightarrow \mathsf{E}(\neg 0 \wedge \langle s\rangle\top)$.

  (b) Axioms for scalar-vector product:

  (b1) $\langle s\rangle(\varphi \vee \psi) \leftrightarrow \langle s\rangle\varphi \vee \langle s\rangle\psi$

  (b2) $\langle s\rangle\varphi \rightarrow \mathsf{E}\varphi$

  (b3) $\mathsf{E}(\neg 0 \wedge \langle s\rangle\top) \rightarrow (\langle s\rangle\neg\varphi \leftrightarrow (\langle s\rangle\top \wedge \neg\langle s\rangle\varphi))$

  (b4) $\langle s\cdot t\rangle\varphi \leftrightarrow \langle s\rangle\langle t\rangle\varphi$

  (b5) $\langle t\rangle(\varphi \oplus \psi) \leftrightarrow (\langle t\rangle\varphi \oplus \langle t\rangle\psi)$

  (b6) $\langle s+t\rangle\mathsf{i} \leftrightarrow \langle s\rangle\mathsf{i} \oplus \langle t\rangle\mathsf{i}$

Dynamic vector logic

- (c) Further laws for field addition and multiplication:

  (c1) $\langle 0\rangle\varphi \leftrightarrow (0 \wedge \mathsf{E}\varphi)$

  (c2) $\langle 1\rangle\varphi \leftrightarrow \varphi$

  (c3) $\langle -s\rangle\varphi \leftrightarrow \langle -\rangle\langle s\rangle\varphi$

  (c4) $\langle s \cdot t\rangle\varphi \leftrightarrow \langle t \cdot s\rangle\varphi$

  (c5) $(\mathsf{i} \wedge \langle s^{-1}\rangle\mathsf{j}) \leftrightarrow (\mathsf{E}(\neg 0 \wedge \langle s\rangle\top) \wedge \mathsf{E}(\mathsf{j} \wedge \langle s\rangle\mathsf{i}))$

  (c6) $(\mathsf{E}(\neg 0 \wedge \langle s\rangle\top) \wedge \mathsf{i}) \rightarrow \langle s^{-1}\rangle\langle s\rangle\mathsf{i}$

- The additional rules of inference for DVL over LCG are as follows.

  Necessitation Rules: $\frac{\neg\varphi}{\neg\langle s\rangle\varphi}$ for each term s

  Extra Witness Rule: $\frac{\langle s\rangle(\mathsf{j} \wedge \varphi) \rightarrow \theta}{\langle s\rangle\varphi \rightarrow \theta}$, where the nominal j does not occur in φ or θ.

  Substitution Rule: Nominals in provable formulas can be replaced by point formulas.

Dynamic vector logic

(a) Axioms for definedness of terms:

(a1) $\mathsf{E}\langle s + t\rangle\top \leftrightarrow \mathsf{E}\langle s\rangle\top \wedge \mathsf{E}\langle t\rangle\top$

(a2) $\mathsf{E}\langle s \cdot t\rangle\top \leftrightarrow \mathsf{E}\langle s\rangle\top \wedge \mathsf{E}\langle t\rangle\top$

(a3) $\mathsf{E}\langle -s\rangle\top \leftrightarrow \mathsf{E}\langle s\rangle\top$

(a4) $\mathsf{E}\langle s^{-1}\rangle\top \leftrightarrow \mathsf{E}(\neg 0 \wedge \langle s\rangle\top)$.

(a4): If $\mathsf{E}(\neg 0 \wedge \langle s\rangle\top)$ is true somewhere, then the value of $s \neq 0$. And then the value of s^{-1} is defined.

Dynamic vector logic

(b) Axioms for scalar-vector product:

(b1) $\langle s\rangle(\varphi \vee \psi) \leftrightarrow \langle s\rangle\varphi \vee \langle s\rangle\psi$

(b2) $\langle s\rangle\varphi \rightarrow \mathsf{E}\varphi$

(b3) $\mathsf{E}(\neg\mathsf{0} \wedge \langle s\rangle\top) \rightarrow (\langle s\rangle\neg\varphi \leftrightarrow (\langle s\rangle\top \wedge \neg\langle s\rangle\varphi))$

(b4) $\langle s \cdot t\rangle\varphi \leftrightarrow \langle s\rangle\langle t\rangle\varphi$

(b5) $\langle t\rangle(\varphi \oplus \psi) \leftrightarrow (\langle t\rangle\varphi \oplus \langle t\rangle\psi)$

(b6) $\langle s + t\rangle\mathsf{i} \leftrightarrow \langle s\rangle\mathsf{i} \oplus \langle t\rangle\mathsf{i}$

The identity $a \bullet (x + y) = a \bullet x + a \bullet y$ underlies (b5)

$(\langle s\rangle\varphi \oplus \langle t\rangle\varphi) \rightarrow \langle s + t\rangle\varphi$ is not valid, but (b6) holds.

Dynamic vector logic

(c) Further laws for field addition and multiplication:

(c1) $\langle 0 \rangle \varphi \leftrightarrow (0 \wedge \mathsf{E}\varphi)$

(c2) $\langle 1 \rangle \varphi \leftrightarrow \varphi$

(c3) $\langle -s \rangle \varphi \leftrightarrow \langle - \rangle \langle s \rangle \varphi$

(c4) $\langle s \cdot t \rangle \varphi \leftrightarrow \langle t \cdot s \rangle \varphi$

(c5) $(\mathsf{i} \wedge \langle s^{-1} \rangle \mathsf{j}) \leftrightarrow (\mathsf{E}(\neg 0 \wedge \langle s \rangle \top) \wedge \mathsf{E}(\mathsf{j} \wedge \langle s \rangle \mathsf{i}))$

(c6) $(\mathsf{E}(\neg 0 \wedge \langle s \rangle \top) \wedge \mathsf{i}) \rightarrow \langle s^{-1} \rangle \langle s \rangle \mathsf{i}$

(c6): if the value of $s \neq 0$, then the value of $s^{-1}s$ is 1.

Derivable formulas in DVL

(1) $\langle (s+t)+u\rangle\varphi \leftrightarrow \langle s+(t+u)\rangle\varphi$

(2) $\langle s+t\rangle\varphi \leftrightarrow \langle t+s\rangle\varphi$

(3) $\langle s+0\rangle\varphi \leftrightarrow \langle s\rangle\varphi$

(4) $\langle s+(-s)\rangle\varphi \leftrightarrow \langle 0\rangle\varphi$

(5) $\langle (s\cdot t)\cdot u\rangle\varphi \leftrightarrow \langle s\cdot(t\cdot u)\rangle\varphi$

(6) $\langle s\cdot 1\rangle\varphi \leftrightarrow \langle s\rangle\varphi$

(7) $\langle s\cdot(t+u)\rangle\varphi \leftrightarrow \langle s\cdot t+s\cdot u\rangle\varphi$

(8) $\mathsf{E}(\neg 0 \wedge \langle s\rangle\top) \rightarrow (\langle s^{-1}\cdot s\rangle\varphi \leftrightarrow \langle 1\rangle\varphi)$.

Dynamic vector logic

Theorem. The proof calculus DVL is sound and complete for validity in vector models.

Idea: Form a field from terms and adjust the construction of the proof of completeness of the logic of commutative groups.

Further research directions

- Can we express properties/theorems of linear algebra in our logic.
- Completeness for the vector logic with the modality C. (More on this will be in the second part!)
- McKinsey-Tarski like results: completeness with respect to particular groups and vector spaces.
- Logics of other structures such as modules.
- In Dynamic Vector Logic we can also fix a field F, e.g., $\mathbb{R}$.

Further research directions

- Logic of linear transformations. We can have terms F varying over linear transformation and hence formulas $\langle F\rangle\varphi$. Then

  $$\langle F\rangle(\varphi \oplus \psi) \leftrightarrow (\langle F\rangle\varphi \oplus \langle F\rangle\psi)$$

  should hold.

  Given a basis, linear transformations are given by matrices, so we can put matrices inside our modalities, and state axioms such as the following:

  $$\langle \mathbf{M_1}\rangle\langle \mathbf{M_2}\rangle\varphi \ \leftrightarrow\ \langle \mathbf{M_1} \times \mathbf{M_2}\rangle\varphi$$

Thank you and over to Johan!

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Modal Structures in Groups and Vector Spaces

Part II

Johan van Benthem & Nick Bezhanishvili

LIRA/LLAMA seminar, ILLC, 9 March 2023

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Defining Subsets in Groups, 1

Given a group and a valuation for proposition letters, modal formulas in our LCG language define an algebra of subsets with the Boolean operations plus product and inverse operations.

Z x Z **V(p) = {(0, 1), (1, 0)}**

-p: {(-1, 0), (0, -1)} **p (+) p**: {(0, 2), (1, 1), (2, 0)}

p (+) –p: {(-1, 1), (0, 0), (1, -1)}

(p (+) –p) & ¬0: {(-1, 1), ((1, -1)} **A**

(p (+) A) & ¬p: {(-1, 2), (2, -1)} **B**

(B (+) B) & (p (+) p): **{(1, 1)}**

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Defining Subsets in Groups, 2

Now we can define each point on the diagonal.

Using these again, define all pairs {(m, k), (k, m)}

by iterated sums with **A** - dropping earlier-defined points

Fact The definable sets are all finite and cofinite reflective sets.

X is **reflective** if for every (m, n) in X: (n, m) is in X.

Proof (a) All finite reflective sets are definable.

(b) The finite and coffinite reflective sets are closed under the operations of our algebra.

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Defining Logical Notions, 1

φ (+) ψ behaves like a linear/categorial logic-style product conjunction (**p**, **p (+) p**, … are all different)

It has a natural binary inverse operation $\varphi \rightarrow \psi$ defined as {x | for all y |= φ: x + y |= ψ}

basic valid principles: (a) **φ (+) ($\varphi \rightarrow \psi$)** |= ψ

(b) If X **(+)** φ |= ψ, then X |= $\varphi \rightarrow \psi$

Two interesting interpretations: **Minkowski operations**, **substructural conjunction plus implication**

Mathematical Morphology connects to logic

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Defining Logical Notions, 2

Fact In LCG, $\varphi \to \psi$ is definable as $\neg(-\varphi\ (+)\ \neg\psi)$

Key point: z = x + y is equivalent to y = z + -x.

similar shift in Relational Algebra for arrow composition, using converse:

left implication R\S for binary relations definable as $\neg(R^{conv} ; -S)$

Question for Modal Information Logic

Can we replace the implication inverse to **<sup>**

[Søren's seminar] using some informationally well-motivated

inverse operation on preorder/poset models?

Modal Fixpoint Logic of Closure, 1

The subgroup **Cφ** generated by a set defined by φ

can be defined by a smallest fixed-point formula

μp. (0 v φ v <–>p v (p (+) p))

Theorem. The modal logic of groups plus linear closure over general relational models is axiomatized by LCG plus

- $(0 \vee \varphi \vee \langle - \rangle C\varphi \vee C\varphi \oplus C\varphi) \rightarrow C\varphi$
- if $\vdash 0 \rightarrow \alpha$, $\vdash \varphi \rightarrow \alpha$, $\vdash \langle - \rangle\alpha \rightarrow \alpha$ and $\vdash \alpha \oplus \alpha \rightarrow \alpha$, then $\vdash C\varphi \rightarrow \alpha$.

Proof Adapt completeness proof for PDL to binary modalities.

Modal Fixpoint Logic of Closure, 2

Open problem Completeness for functional relational models?

Open problem What about completeness on group models?

technical aside plus curious observation

The closure modality **C**φ is monotone, but **not distributive** over disjunction or conjunction: **neighborhood modality**.

Yet the PDL completeness proof adapts without distribution.

Fact The PDL Kleene star modality **[*]**φ distributes over conjunction: but this is **derivable** from just fixed-point proof principles

Logic of Dependence in Vector Spaces, 1

Dependence $D_X\, y$ in vector spaces:

y is a **linear combination** of vectors in X

Basic law of **Steinitz Exchange** $D_{X,\, z}\, y \;\to\; D_X\, y \lor D_{X,\, y}\, z$

Proof y = **a**.X + b.z: two cases b = 0, b ≠ 0

Note: both sets of vectors X and single vectors y, z (nominals)

Fact Steinitz fails in group models:

(Z x Z, +) X = {(1, 1), (2, 1) (= z)}, y = (5, 3).

(2, 1) not definable from {(1, 1), (5, 3)}: unsolvable equations

We need (some) presence of **division**

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Logic of Dependence in Vector Spaces, 2

Fixed-point definition for linear dependence **Dφ**:

μp. (φ v mp v (p (+) p))

Here **mψ** defines the set of all multiples of vectors in the set ψ

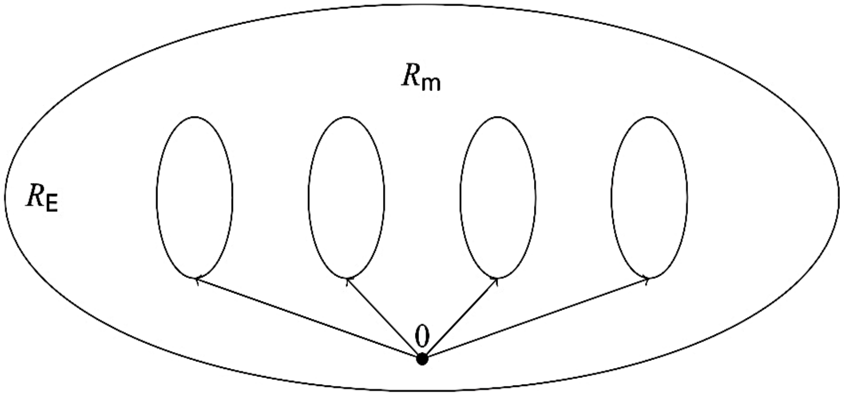
Theorem The pure logic of vector multiples is axiomatized by

1. $\mathsf{m}(\varphi \vee \psi) \leftrightarrow \mathsf{m}\varphi \vee \mathsf{m}\psi$
2. $\varphi \to \mathsf{m}\varphi$
3. $\mathsf{mm}\varphi \to \mathsf{m}\varphi$
4. $(\varphi \wedge \neg 0 \wedge \mathsf{m}(\neg 0 \wedge \psi)) \to \mathsf{m}(\psi \wedge \mathsf{m}\varphi)$
5. $\mathsf{m}(0 \wedge \varphi) \to \varphi$
6. $\mathsf{E}\varphi \to \mathsf{E}(0 \wedge \mathsf{m}\varphi)$
7. $\mathsf{m}\varphi \to \mathsf{E}\varphi$

More Modal Logic of Vector Multiples

What axioms enforce via

frame correspondence



connects to Pin Logic: The largest non-tabular logic below S5, which does have the Finite Model Property

Open problem Axiomatize the complete logic of additive structure (LCG) plus linear dependence in vector spaces.

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Relations to LFD and Other Dependence Logics

State space S, variables map states s, t to values in their ranges

variable y **depends** on set of variables X: **if s $=_X$ t, then s $=_y$ t**

Equivalent to definability: **y(s) = F(X(s))** for some function **F** on values

But: **independence** in vector spaces is negation of dependence

Independence in LFD more complex :

fixing X-values gives no information about y-value.

Difference: **object-level** vs. lifted **function-level** dependencies

Open problem Connect our dependence logics with LFD

Matroids and Independence Logics

Matroid Finite family **F** of finite sets, containing $\varnothing$, closed under subsets, and if A, B $\in$ **F**, |A| < |B|, there is a b $\in$ B s.t. A U {b} $\in$ **F**.

Abstract notion of independent sets. Also non-vector models!

Modal analogue Independence is now a **predicate** **I**φ

analogy: global modalities

The paper gives just a bunch of valid principles,

but leaves an **open problem**

Axiomatize modal independence logic on vector models

PS Logics of Dependence plus Independence

Pure modal logic of independence may be challenging:

e.g., the cardinality condition in matroids seems to call

for additional **numerical comparison modalities**

from the literature on **graded modal logics**

Alternative Axiomatize the joint logic of D and I

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Matroids and (In-)Dependence Logics

Can also do Matroid Theory via dependence closure notion

Abstract connections [results in paper]:

Independence predicates [matroids via definable sets]

induce dependence predicates satisfying LFD + Steinitz,

and there is also a converse construction

Open problem Is there also a two-way translation between modal logics for dependence and for independence in vector models?

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Excursion: Infinite Matroids and Modal Logic

Recent extension of Matroid Theory to **infinite matroids**

All conditions for the finite case plus

Take any subset X of the total domain of objects. Any independent set I contained in X can be extended to a maximally independent set *among the subsets of* X

this is a wellfoundedness condition familiar from modal logic

Open problem

Find a connection between infinite matroid theory and modal logic

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Where Our Approach May Lead

Contact modal logic linear algebra [if it works]:

compare with benefits **modal logic and topology**:

- Import topology into understanding logical systems
- Develop more abstract forms of topology suggested

  by abstract models for topological modal logics
- Find fragments with low computational complexity

Some Interesting Challenges

Modal logics have been especially successful with

describing **tree-like** structures

Even topological semantics works via Alexandrov tree topologies

But are vector spaces like that?

Our abstract relational models do admit of tree unraveling

(in the generalized style of the Guarded Fragment etc.)

But we would need good **representation theorems**

In line with this: no general insight yet on **decidability**

or **SAT complexity** of the logics presented here