Games with Different Decision Criteria

Paolo Galeazzi

November 17, 2022
Intro

- Michael Franke
- Johannes Marti
- Mathias Madsen
- Alessandro Galeazzi
Outline

General idea: Games with different decision criteria

▶ From an evolutionary point of view
▶ From a game-theoretic point of view
▶ From an epistemic point of view
From an evolutionary point of view

- Single-game model
- Evolution of simple behavior

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From an evolutionary point of view

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Replicator dynamics [Taylor and Jonker, 1978]:

$$\dot{p}_i = p_i (u(a_i, \sigma) - u(\sigma, \sigma))$$

where:
- $u$ utility/payoff/fitness
- $a_i$ action of type $i$
- $\sigma$ action mix in the population
From an evolutionary point of view

By focusing on expressed behavior and neglecting the underlying mechanism, behavioral ecologists unwittingly adopt the behavioral gambit, extending the phenotypic gambit beyond its accepted remit.
From an evolutionary point of view

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Natural environments are so complex, dynamic, and unpredictable that natural selection cannot possibly furnish an animal with an appropriate, specific behavior pattern for every conceivable situation it might encounter. Instead, we should expect animals to have evolved a set of psychological mechanisms which enable them to perform well on average across a range of different circumstances.

[Fawcett et al., 2012]
From an evolutionary point of view

The model:
- The environment: a set $G$ of symmetric two-player games
- The types: decision criteria, mechanisms that produce behavior in each game
From an evolutionary point of view

The model:

- The environment: a set $\mathcal{G}$ of symmetric two-player games
- The types: decision criteria, mechanisms that produce behavior in each game

Example:

- The environment:

<table>
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<tr>
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<td>2,2</td>
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<td>3,0</td>
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<td></td>
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- The types: $\mathcal{T} \subseteq \{I, II\}^{\left|\mathcal{G}\right|} = \{I, II\} \times \{I, II\}$
From an evolutionary point of view

Decision criteria as types:

Decision criteria:

\[ d: \text{Utilities} \times \text{Beliefs} \rightarrow \text{Actions} \]

where

- \[ u: Z \rightarrow \mathbb{R} \] utility function
- \[ B \subseteq \Delta(S) \] belief

Decision problem:

\[(S, A, Z, c)\]

where

- \[ S \] set of states of the world
- \[ A \] set of actions
- \[ Z \] set of outcomes
- \[ c: S \times A \rightarrow Z \] outcome function
From an evolutionary point of view

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<th>1/3</th>
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<td>0</td>
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A game is a decision problem:

- \( S = A^{-i} = \{I, II\} \)
- \( A_i = \{I, II\} \)
- \( Z = \{(3, 3), (0, 2), (2, 0), (2, 2)\} \)
- \( c((I, I)) = (3, 3), \ c((I, II)) = (0, 2), \ c((II, I)) = (2, 0), \ c((II, II)) = (2, 2) \)
- \( u_i((3, 3)) = 3, \ u_i((0, 2)) = 0, \ u_i((2, 0)) = 2, \ u_i((2, 2)) = 2 \)
- \( B = \{p \in \Delta(S) : 0 \leq p(I) \leq 1/3\} \)
From an evolutionary point of view

Decision criteria as types:

<table>
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<tr>
<td>▶ Expected utility maximization</td>
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<tr>
<td>[ a^* \in \arg\max_{a\in A} E_p[u</td>
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From an evolutionary point of view

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Where

$$E_p[u|a] := \sum_{s \in S} u(c(s, a))p(s)$$

and

$$r_R(a, p) := E_p \left[ \max_{a' \in A} u(c(s, a')) - u(c(s, a)) \right] = \sum_{s \in S} p(s) \left( \max_{a' \in A} u(c(s, a')) - u(c(s, a)) \right)$$
From an evolutionary point of view

Decision criteria as types: example.

\[
\begin{array}{c|cc}
 & 1/3 & 2/3 \\
\hline
0 & 0 & 1 \\
SH & I & II \\
\hline
I & 3,3 & 0,2 \\
II & 2,0 & 2,2 \\
\end{array}
\]

- Expected utility maximization (+ principle of insufficient reason):
  \[
  E[u|I] = 3 \cdot 1/6 = 0.5 \\
  E[u|II] = 2 \cdot 1/6 + 2 \cdot 5/6 = 2
  \]

- Maxmin expected utility:
  \[
  \min_{p \in B} E_p[u|I] = \min\{1, 0\} = 0 \\
  \min_{p \in B} E_p[u|II] = \min\{2, 2\} = 2
  \]
From an evolutionary point of view

Decision criteria as types:

Less classic decision criteria:

- Distribution-regret minimization

\[ a^* \in \arg\min_{a \in A} \max_{p \in B} r_D(a, p) \]

- Altruistic expected utility maximization

\[ a^* \in \arg\max_{a \in A} E_p[\tilde{u} | a] \]

- Random type: pick an action at random

- Action-I type: \( d(u, B) = I \) for all \( u \) and \( B \)

- ...
From an evolutionary point of view

Decision criteria as types:

Less classic decision criteria:

- Distribution-regret minimization

\[ a^* \in \arg\min_{a \in A} \max_{p \in B} r_D(a, p) \]

- Altruistic expected utility maximization

\[ a^* \in \arg\max_{a \in A} E_p[\tilde{u}|a] \]

- Random type: pick an action at random

- Action-1 type: \( d(u, B) = l \) for all \( u \) and \( B \)

- ...

Where

\[ r_D(a, p) := \max_{a' \in A} E_p\left[u|a'\right] - E_p[u|a] = \max_{a' \in A} \sum_{s \in S} p(s)u(c(s, a')) - \sum_{s \in S} p(s)u(c(s, a)) \]

and

\[ \tilde{u}(a, a') := u_1(a, a') + u_2(a, a') = u(a, a') + u(a', a) \]
From an evolutionary point of view

What we have done so far:

▶ Enrich the environment: from single-game to multi-game model

Not investigated very much in the literature in evolutionary game theory:

▶ [Zollman, 2008]: multi-game with three games only (Nash bargaining game, ultimatum game, a hybrid of the two), no full-fledged decision criteria

▶ [Klein et al., 2018]: decision criteria as types (ABM, not EGT), no multi-game environment

▶ [Lecouteux, 2015]: team reasoning as underlying mechanism, but no multi-game environment

▶ Evolution of preferences (i.e., utility functions): preferences as underlying mechanisms, but no multi-game environment (e.g., [Dekel et al., 2007, Robson and Samuelson, 2011])
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What we have done so far:

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From an evolutionary point of view

Analytic results:

- **Proposition 1.** Let $\mathcal{G}$ be the class of $2 \times 2$ symmetric games $G = (a, b, c, d)$ generated by i.i.d. sampling $a, b, c, d$ from a set of values with at least three elements in the support. Then, distribution-regret minimization strictly dominates maxmin in the resulting multi-game.

- **Corollary 1.** Let $\mathcal{G}$ be as in Proposition 1. The unique evolutionarily stable state in a population of maximinimizers and distribution-regret minimizers is a monomorphic population of regret minimizers.

[Galeazzi and Franke, 2017]
From an evolutionary point of view

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[Galeazzi and Franke, 2017]
From an evolutionary point of view

Simulation results:

<table>
<thead>
<tr>
<th>Table 1</th>
<th>Number of times $EU$ strictly dominates all other criteria for all combinations of $m$ (in the rows), $n$ (in the columns), and $\overline{v}$ ($\overline{v} = 50$ in Table 1a, and $\overline{v} = 100$ in Table 1b)</th>
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</tbody>
</table>

- $\{0, \ldots, \overline{v}\}$ the set of randomly drawn fitness values
- $n$ the number of actions in the games
- $m$ the number of points in the belief set $B$

[Galeazzi and Galeazzi, 2020]
From a game-theoretic point of view

- Gap in the literature: games with homogeneous vs heterogeneous criteria
- Homogeneous criteria:
  - [Lo, 1996]: maxmin expected utility
  - [Klibanoff, 1996]: maxmin expected utility
  - [Marinacci, 2000]: Choquet expected utility
  - [Kajii and Ui, 2005]: maxmin expected utility
  - [Renou and Schlag, 2010]: realization-regret minimization
  - [Halpern and Pass, 2012]: realization-regret minimization
  - ...
From a game-theoretic point of view

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- Heterogeneous criteria:
  - [Epstein, 1997]: introduces the concepts of rationalizability, (iterated) dominance and equilibrium for general preferences (i.e., utility functions) on acts. Based on [Epstein and Wang, 1996], more on this later.
From a game-theoretic point of view

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- Heterogeneous criteria:
  - [Epstein, 1997]: introduces the concepts of rationalizability, (iterated) dominance and equilibrium for general preferences (i.e., utility functions) on acts. Based on [Epstein and Wang, 1996], more on this later.
  - Team reasoning: [Bacharach, 1999, Lecouteux, 2018]
From a game-theoretic point of view

Two-player Bertrand competition. Each firm $i$ produces the same commodity at a cost $c_i$, and sell it at a price of $p_i$. The firm that chooses the lowest price captures the whole market, pocketing a profit of $p_i - c_i$, or $(p_i - c_i)/2$ in the case of a tie. Hence:

$$u_1(c_1, c_2, p_1, p_2) = \begin{cases} 
  p_1 - c_1 & p_1 < p_2 \\
  (p_1 - c_1)/2 & p_1 = p_2 \\
  0 & p_1 > p_2 
\end{cases}$$

where $c_1, c_2$ are the types/private information of the players and $p_1, p_2$ are the actions of the players, with $c_i \in C_i := \{0, 0.1, \ldots, 1\}$ and $p_i \in P_i := \{0, 0.1, \ldots\}$. 
From a game-theoretic point of view

Suppose firm 1 is regret minimizer and firm 2 is maxmin, and $B_i|c_i = \Delta(C_{-i})$ for all $c_i$. 
Suppose firm 1 is regret minimizer and firm 2 is maxmin, and 
\( B_i|c_i = \Delta(C_{\neg i}) \) for all \( c_i \).

**Equilibrium.** An equilibrium is then a pair of strategies \((\sigma_1, \sigma_2)\) with \( \sigma_i : C_i \to P_i \) such that:

1. \( \forall c_1 \in C_1, \sigma_1(c_1) \in \arg \min_{p_1} \max_{q \in B_1|c_1} r_R(p_1, c_1, q) \)
2. \( \forall c_2 \in C_2, \sigma_2(c_2) \in \arg \max_{p_2} \min_{q \in B_2|c_2} E_q[u_2|p_2, c_2] \)
From a game-theoretic point of view

Suppose firm 1 is regret minimizer and firm 2 is maxmin, and $B_i|c_i = \Delta(C_{-i})$ for all $c_i$.

**Equilibrium.** An equilibrium is then a pair of strategies $(\sigma_1, \sigma_2)$ with $\sigma_i : C_i \rightarrow P_i$ such that:

1. $\forall c_1 \in C_1, \sigma_1(c_1) \in \arg\min_{p_1} \max_{q \in B_1|c_1} r_R(p_1, c_1, q)$
2. $\forall c_2 \in C_2, \sigma_2(c_2) \in \arg\max_{p_2} \min_{q \in B_2|c_2} E_q[u_2|p_2, c_2]$

Where

$$E_q[u_2|p_2, c_2] := \sum_{c_1 \in C_1} q(c_1)u_2(c_1, c_2, \sigma_1(c_1), p_2))$$

and

$$r_R(p_1, c_1, q) := E_q \left[ \max_{p_1'} u_1(c_1, c_2, p_1', \sigma(c_2)) - u_1(c_1, c_2, p_1, \sigma(c_2)) \right]$$

$$= \sum_{c_2 \in C_2} q(c_2) \left( \max_{p_1'} u_1(c_1, c_2, p_1', \sigma(c_2)) - u_1(c_1, c_2, p_1, \sigma(c_2)) \right)$$
From a game-theoretic point of view

The pair of strategies

\[ \sigma_R(c_1) = \begin{cases} 
\frac{1.1 + c_1}{2} & \text{c_1 odd} \\
\frac{1.1 + c_1}{2} - 0.05 & \text{c_1 even} 
\end{cases} \]

and

\[ \sigma_M(c_2) = \begin{cases} 
0.4 & c_2 < 0.4 \\
0.5 & c_2 = 0.4 \\
c_2 + 0.1 & 0.4 < c_2 < 1 \\
1 & c_2 = 1 
\end{cases} \]

constitute an equilibrium of the game.

- In equilibrium, firm 1 or firm 2 gets positive profit
- The same pricing strategies do not constitute a single-criterion equilibrium
- \((\sigma_R, \sigma_R)\) is a regret-equilibrium, but \((\sigma_M, \sigma_M)\) is not a maxmin-equilibrium
From an epistemic point of view

- [Epstein and Wang, 1996]
- [Di Tillio, 2008]
- [Bjorndahl et al., 2017]
From an epistemic point of view


Savage approach (single-agent decision problem):

▶ Primitives: states of the world $S$, outcomes $Z$, acts $f \in Z^S$, preferences over acts
From an epistemic point of view


Savage approach (single-agent decision problem):

- Primitives: states of the world $S$, outcomes $Z$, acts $f \in Z^S$, preferences over acts
- Utility and probabilistic beliefs are derived from preferences
From an epistemic point of view

Type spaces (interactive decision problem):
- Primitives: states of nature $A_{-i}$ (i.e., parameters of the game and actions of the opponents), player $i$’s types $T_i$, player $i$’s belief function $\beta_i : T_i \rightarrow \Delta(A_{-i} \times T_{-i})$
From an epistemic point of view

Type spaces (interactive decision problem):

- Primitives: states of nature $A_{-i}$ (i.e., parameters of the game and actions of the opponents), player $i$’s types $T_i$, player $i$’s belief function $\beta_i : T_i \to \Delta(A_{-i} \times T_{-i})$

- Type spaces generate hierarchies of interactive probabilistic beliefs, e.g.

$$t_i^l \mapsto (1/3 t_j^l, 2/3 t_j^{ll}) \quad t_j^l \mapsto (1 t_i^l)$$

$$t_i^{ll} \mapsto (3/4 t_j^l, 1/4 t_j^{ll}) \quad t_j^{ll} \mapsto (2/5 t_i^l, 3/5 t_i^{ll})$$

$$(t_i^l, t_j^l) \models B_i^{1/3} I_j \quad (t_i^l, t_j^l) \models B_i^{2/3} (II_j \land B_j^{3/5} II_i)$$

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From an epistemic point of view

Type spaces (interactive decision problem):

- Primitives: states of nature $A_{-i}$ (i.e., parameters of the game and actions of the opponents), player $i$’s types $T_i$, player $i$’s belief function $\beta_i : T_i \rightarrow \Delta(A_{-i} \times T_{-i})$
- Type spaces generate hierarchies of interactive probabilistic beliefs, e.g.

\[
\begin{align*}
  t_i^l &\mapsto (1/3t_j^l, 2/3t_j^{ll}) & t_j^l &\mapsto (1t_i^l) \\
  t_i^{ll} &\mapsto (3/4t_j^l, 1/4t_j^{ll}) & t_j^{ll} &\mapsto (2/5t_i^l, 3/5t_i^{ll}) \\
  (t_i^l, t_j^l) &| B_i^{1/3} I_j & (t_i^l, t_j^l) &| B_i^{2/3}(II_j \land B_j^{3/5} II_i)
\end{align*}
\]

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<td>1/4, 3/5</td>
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- Loosely, $B_i^{1/3} I_j \in \Delta(A_j)$, $B_i^{2/3}(II_j \land B_j^{3/5} II_i) \in \Delta(A_j \times \Delta(A_i))$, ...

...
From an epistemic point of view

Type spaces (interactive decision problem):

- Primitives: states of nature $A_{-i}$ (i.e., parameters of the game and actions of the opponents), player $i$’s types $T_i$, player $i$’s belief function $\beta_i : T_i \rightarrow \Delta(A_{-i} \times T_{-i})$

- Type spaces generate hierarchies of interactive probabilistic beliefs, e.g.

\[
\begin{align*}
  t_i^I & \mapsto (1/3 t_j^I, 2/3 t_j^{II}) & t_j^I & \mapsto (1 t_i^I) \\
  t_i^{II} & \mapsto (3/4 t_j^I, 1/4 t_j^{II}) & t_j^{II} & \mapsto (2/5 t_i^I, 3/5 t_i^{II})
\end{align*}
\]

\[
(t_i^I, t_j^I) \models B_i^{1/3} I_j \quad (t_i^I, t_j^I) \models B_i^{2/3} (II_j \land B_j^{3/5} II_i)
\]

<table>
<thead>
<tr>
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\[
\begin{array}{|c|c|c|}
\hline
& t_i^I & t_j^{II} \\
\hline
t_i^I & 1/3, 1 & 2/3, 2/5 \\
t_i^{II} & 3/4, 0 & 1/4, 3/5 \\
\hline
\end{array}
\]

- Loosely, $B_i^{1/3} I_j \in \Delta(A_j)$, $B_i^{2/3} (II_j \land B_j^{3/5} II_i) \in \Delta(A_j \times \Delta(A_i))$, ...

- Probabilistic beliefs are taken as primitive and not derived from behavior, i.e., preferences
From an epistemic point of view

[Epstein and Wang, 1996] and [Di Tillio, 2008] preference structures:

- Quite similar, technical differences not relevant for our purposes here
- Savage approach into type spaces

Primitives: states of nature

\( A_{-i} \) (i.e., parameters of the game and actions of the opponent), player \( i \)'s types \( T_i \), player \( i \)'s "preference" function \( \theta_i : T_i \to \Pi(A_{-i} \times T_{-i}) \)

\( \Pi \) vs \( \Delta \):

\( \Pi(X) = P(F(X)) \)

\( F(X) := Z \) \( X \) is the set of acts over \( X \) for given outcome set \( Z \)

\( P(F(X)) \) is the set of all preference relations over acts over \( X \)

Preference structures generate hierarchies of interactive preference relations (see next slide)
From an epistemic point of view

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▸ Quite similar, technical differences not relevant for our purposes here
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- $\Pi$ vs $\Delta$: $\Pi(X) = \mathcal{P}(F(X))$
  - $F(X) := Z^X$ is the set of acts over $X$ for given outcome set $Z$
From an epistemic point of view

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$S = A_j = \{ I, II \}$

$Z = \{(3, 3), (0, 2), (2, 0), (2, 2)\}$

Hierarchies of preference relations:

1. $\pi_1 \in \Pi(A_j)$, i.e., preference relation over acts $Z^{A_j}$
2. $\pi_2 \in \Pi(A_j \times \Pi(A_i))$, i.e., preference relation over acts $Z^{A_j \times \Pi(A_i)}$
3. ...

Remark. But where are all those acts? E.g., the act $f(x) = (2, 2)$ for all $x \in A_j$
From an epistemic point of view

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- Preference relations are transitive, regret is not

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Table: Regret is not transitive

I \succ II \succ III \succ I

- Choice structures: from hierarchies of interactive preference relations to hierarchies of interactive choice functions

\( C : 2^X \rightarrow 2^X \) s.t. \( C(Y) \subseteq Y \) for all \( Y \subseteq X \)
From an epistemic point of view

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\( \Gamma(X) := C(F(X)) \)

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From an epistemic point of view

Results:

\textit{Theorem 1.} $\Omega_i \simeq \Gamma(A_j \times \Omega_j)$ and $\Omega_j \simeq \Gamma(A_i \times \Omega_i)$.

\textit{Theorem 2.} For every choice structure $\mathcal{X}$ there is a unique morphism of choice structures $\nu : \mathcal{X} \rightarrow \mathcal{U}$ from $\mathcal{X}$ to the universal choice structure $\mathcal{U}$.

\vspace{1em}

\begin{itemize}
  \item which is a complicated way to say that there exists a universal choice structure.
\end{itemize}
From an epistemic point of view

Results:

Theorem 1. \( \Omega_i \simeq \Gamma(A_j \times \Omega_j) \) and \( \Omega_j \simeq \Gamma(A_i \times \Omega_i) \).

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▶ which is a complicated way to say that there exists a universal choice structure.

Theorem 3. Assume that \( A_i \) and \( A_j \) are finite. Then the uncertainty spaces \( \Omega'_i \) and \( \Omega'_j \) of all preference hierarchies are isomorphic to two subspaces of the spaces of all choice hierarchies \( \Omega_i \) and \( \Omega_j \) respectively.

▶ which is a complicated way to say that choice structures embed preference structures.
Conclusion

Thanks for your attention.
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