Point-Set Neighborhood Logic

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NihiL/LIRa seminar, ILLC

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World-Team Neighborhood Logic

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Background: my view

Point-Set Neighborhood Logic

Hilbert system and sequent calculus

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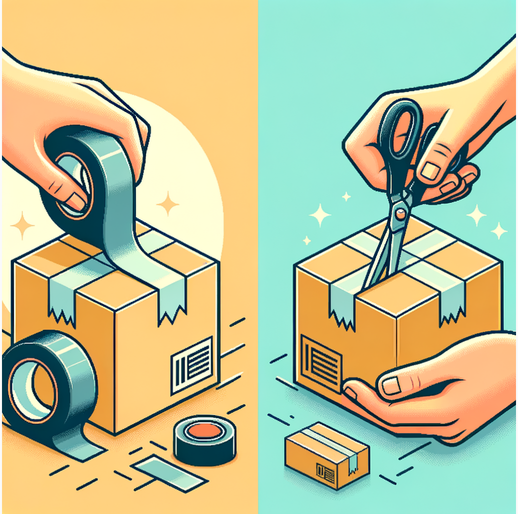
Conclusions and future work

Background: my view

Bundling and unbundling



Bundling and unbundling in modal and non-classical logics

Bundling: take a construction as a modality

- Temporal Logic e.g., $CTL : AF\phi, ATL : \langle\langle A \rangle\rangle \phi U \psi$
- Epistemic Logic of *Know-wh*:

$$\mathrm{Kw}\phi := \mathrm{K}\phi \vee \mathrm{K}\neg\phi, \mathrm{Kv}d := \exists x \mathrm{K}(x \approx d), \mathrm{Kh}\phi := \exists \sigma \mathrm{K}[\sigma]\phi$$

- Bundled fragments of FOML (Wang 2017 -):

$$\alpha ::= P(x_1, \dots, x_n) \mid \neg\alpha \mid \alpha \wedge \alpha \mid \Box\alpha \mid \forall x \Box\alpha \mid \exists x \Box\alpha \mid \Box\forall x \alpha \mid \Box\exists x \alpha$$

- Other applications: Modal syllogistic, Deontic logic, Contingency logic, Group knowledge ...

Usual advantages: capture the concepts as a whole; balancing expressivity and complexity... E.g., Normal modalities are bundles too! $\Box\phi := \forall y(xRy \to \phi)$

ESSLLI24 Course:

http://wangyanjing.com/introduction-to-bundled-modalities/

Bundling and unbundling in modal and non-classical logics

Unbundling: break constructions into certain components

- Temporal Logic: CTL^*, ATL^*
- Epistemic Logic of *de re* updates (Cohen, Tang & W. 21):

$$\alpha ::= t \approx t \mid P\vec{t} \mid \neg\alpha \mid \alpha \wedge \alpha \mid [x := t]\alpha \mid \mathsf{K}\alpha \mid [!\alpha]\alpha$$

$$[x := c]\mathsf{K}(x \approx c), [x := c][y := d]\mathsf{K}(M(x, y)), [x := c][!c \approx x]\alpha$$

- Non-classical logic: intuitionistic and intermediate logics as epistemic logic of *knowing how* (Wang^3 21 22)

Advantages: compositional, (sometimes) easier to axiomatize

Bundling or unbundling?

That is the question.

We take a certain neighborhood logic as a case study to demonstrate the use of unbundling.

Not so much about fancy or surprising techniques, but that is the point: making simple things simple!

My personal taste in research:

`Max` (conceptual significance – technical complexity)

Neighborhood Structures

Neighborhood (nbd) frame: $\mathfrak{F} = (W, N)$

- $W \neq \varnothing$, a set of possible worlds;
- $N : W \to 2^{2^W}$, a nbd function.

Nbd Model: $\mathfrak{M} = (\mathfrak{F}, V)$

Different perspectives:

- Technical tools for non-normal modal logic
- As genuine structures or abstractions of finer structures

Monotonic Neighborhood Logic

Monotonic neighborhood logic with a unary operator $\Box$.

$$\mathfrak{M}, w \models \Box\alpha \text{ iff } (\exists X \in N(w))(\forall x \in X)\, \mathfrak{M}, x \models \alpha.$$

There exists a nbd of w has α true everywhere inside.

Some schemes as examples:

invalid	valid
$\Box\alpha \wedge \Box\beta \to \Box(\alpha \wedge \beta)$	$\Box(\alpha \wedge \beta) \to \Box\alpha \wedge \Box\beta$
$\dfrac{\models \phi}{\models \Box\phi}$	$\Box\bot \to \Box\alpha$
	$\dfrac{\models \phi \to \psi}{\models \Box\phi \to \Box\psi}$

Instantial Neighborhood Logic (van Benthem et al. 2017)

Instantial Neighborhood Logic adds instances in the modality where $j \in \mathbb{N}$:

$$\Box(\alpha_1, \ldots, \alpha_j; \alpha_0)$$

$$\mathfrak{M}, w \vDash \Box(\alpha_1, \ldots, \alpha_j; \alpha_0) \text{ iff } \exists X \in N(w) \begin{cases} (\forall x \in X)\, \mathfrak{M}, x \vDash \alpha_0 \\ (\exists x_1 \in X)\, \mathfrak{M}, x_1 \vDash \alpha_1 \\ \vdots \\ (\exists x_j \in X)\, \mathfrak{M}, x_j \vDash \alpha_j \end{cases}$$

There exists a nbd of w that has the following properties:

- α_0 holds everywhere inside, and
- each instance (respectively) holds somewhere inside.

Instantial nbd logic INL (van Benthem et al 2017)

Some invalid schemes:

- $\neg\Box(;\bot)$, $\Box(;\alpha)\wedge\Box(;\beta)\to\Box(;\alpha\wedge\beta)$
- $\dfrac{\vDash\alpha}{\vDash\Box(;\alpha)}$, $\Box(\alpha;\gamma)\wedge\Box(\beta;\gamma)\to\Box(\alpha,\beta;\gamma)$

Some valid schemes:

- $\Box(\alpha_1,\ldots,\alpha_j;\alpha_0)\to\Box(\alpha_1,\ldots,\alpha_j;\alpha_0\vee\eta)$
- $\Box(\alpha_1,\ldots,\alpha_j,\beta;\alpha_0)\to\Box(\alpha_1,\ldots,\alpha_j,\beta\vee\gamma;\alpha_0)$
- $\Box(\alpha_1,\alpha_2,\ldots,\alpha_j;\alpha_0)\to\Box(\alpha_2,\ldots,\alpha_j;\alpha_0)$
- $\Box(\alpha_1,\ldots,\alpha_j;\alpha_0)\to\Box(\alpha_1,\ldots,\alpha_j,\alpha_i;\alpha_0)$ where $i\in\{1,\ldots,j\}$
- $\Box(\alpha_1,\ldots,\alpha_j,\eta;\alpha_0)\to\Box(\alpha_1,\ldots,\alpha_j,\eta\wedge\alpha_0;\alpha_0)$
- $\neg\Box(\alpha_1,\ldots,\alpha_j,\bot;\alpha_0)$
- $\Box(\alpha_1,\ldots,\alpha_j;\alpha_0)\to(\Box(\alpha_1,\ldots,\alpha_j,\delta;\alpha_0)\vee\Box(\alpha_1,\ldots,\alpha_j;\alpha_0\wedge\neg\delta))$

With propositional tautologies and replacement of equivalence, a complete axiomatization of INL is obtained.

Sequent calculus G3inl (Yu 2020)

$$\left(\begin{array}{lll} J = \{-j, \dots, -1\} & K = \{1, \dots, k\} & \\ K^{(l)} = \{y \in K \,|\, l(y) = 0\} & \Omega^l_K = \left\{\beta^i_{l(i)}\right\}^{l(i) \neq 0}_{i \in K} & D = \bigotimes_{i \in K} \{0, 1, \dots, j_i\} \end{array}\right)$$

$$\frac{\left[\alpha_0, \alpha_{-f_l} \Rightarrow \Omega^l_K\right]^{f_l \in J}_{l \in D} \left[\alpha_0 \Rightarrow \beta^{f_l}_0, \Omega^l_K\right]^{f_l \in K^{(l)}}_{l \in D}}{\Pi, \Box(\alpha_1, \dots, \alpha_j; \alpha_0) \Rightarrow \{\Box(\beta^i_1, \dots, \beta^i_{j_i}; \beta^i_0)\}_{i \in K}, \Sigma}$$

- This is $\left(\Box^{j,k,f}_{\langle j_1, \dots, j_k \rangle}\right)$, nbd rule with parameters $j, k, j_1, \dots, j_k, f$.
- It respects the proper sub-formula property (no built-in contraction).
- G3inl is G3cp (in the language of INL) extended by
  all $\left(\Box^{j,k,f}_{\langle j_1, \dots, j_k \rangle}\right)$ where $f : D \to J \cup K$ is adequate:
  i.e., $(\forall l \in D)(f_l \in K$ implies $f_l \in K^{(l)}$, e.g., $l(f_l) = 0)$.

G3inl admits Weakening, Contraction, and Cut, and supports mechanical proof-search. By applying Maehara's method using a splitting version of it, Yu (2020) constructively showed that INL has Lydon Interpolation.

Bundling and unbundling

The bundles behind the semantics:

- The standard semantics for $\Box\alpha$: $\exists X \forall w$
- Instantial neighborhood logic: $\exists X(\overrightarrow{\exists v_i}; \forall w)$

The (weak) completeness of the Hilbert system of INL is based on a normal form argument in van Benthem et al. 2017. The sequent calculus of INL also looks complicated.

Can we do everything much simpler?

What about unbundling INL?

Point-Set Neighborhood Logic

Point-Set Neighborhood Language

A two-sorted modal language with two types of formulas.

Definition (Language $\mathcal{L}^{\mathrm{ps}}(\boxdot, \boxtimes)$)

The language $\mathcal{L}^{\mathrm{ps}}(\boxdot, \boxtimes)$ of *point-formulas* α is defined by the following mutual induction with the *set-formulas* ϕ:

$$\mathcal{L}^{p} \ni \alpha ::= \quad \bot \mid p \mid (\alpha \to \alpha) \mid \boxdot\phi$$
$$\mathcal{L}^{s} \ni \phi ::= \quad \neg\phi \mid (\phi \to \phi) \mid \boxtimes\alpha$$

Note that $\mathcal{L}^{p} \cap \mathcal{L}^{s} = \varnothing$. E.g., $\boxtimes p$ is not a point-formula but $\boxdot \boxtimes p$ is. $\vee$, $\wedge$, $\leftrightarrow$, are defined classically for all formulas. Define $\neg\alpha$ as $\alpha \to \bot$, $\top$ as $\neg\bot$, $\diamond\phi$ as $\neg \boxdot \neg\phi$, and $\otimes\alpha$ as $\neg \boxtimes \neg\alpha$.

$\alpha, \beta, \gamma, \delta, \theta$ are used for point-formulas and $\Gamma, \Delta, \Theta, \Omega, \Upsilon$ for sets/multi-sets of them; $\phi, \psi, \pi, \sigma, \xi$ are used for set-formulas and $\Phi, \Psi, \Pi, \Sigma, \Xi$ for sets/multi-sets of them.

Semantics

Given a nbd model $\mathfrak{M} = \langle W, N, V \rangle$, the satisfaction relation $\vDash$ between a world w and a point-formula α is defined mutually with the relation $\Vdash$ between a set X of worlds and a set-formula ϕ:

$\mathfrak{M}, w \vDash \bot$	$\Leftrightarrow$	*never*
$\mathfrak{M}, w \vDash p$	$\Leftrightarrow$	$w \in V(p)$
$\mathfrak{M}, w \vDash (\alpha \to \beta)$	$\Leftrightarrow$	$\mathfrak{M}, w \nvDash \alpha$ or $\mathfrak{M}, w \vDash \beta$
$\mathfrak{M}, w \vDash \boxdot \phi$	$\Leftrightarrow$	for all $X \in N(w)$: $\mathfrak{M}, X \Vdash \phi$
$\mathfrak{M}, X \Vdash \neg \phi$	$\Leftrightarrow$	$\mathfrak{M}, X \nVdash \phi$
$\mathfrak{M}, X \Vdash (\phi \to \psi)$	$\Leftrightarrow$	$\mathfrak{M}, X \nVdash \phi$ or $\mathfrak{M}, X \Vdash \psi$
$\mathfrak{M}, X \Vdash \boxtimes \alpha$	$\Leftrightarrow$	for all $v \in X$: $\mathfrak{M}, v \vDash \alpha$

An INL formula $\Box(\alpha_1, \ldots \alpha_m; \beta)$ can be viewed as a formula in $\mathcal{L}^{\mathrm{ps}}(\boxdot, \boxtimes)$: $\Diamond(\otimes\alpha_1 \wedge \cdots \wedge \otimes\alpha_m \wedge \boxtimes\beta)$.

Induced semantics for defined connectives and modalities:

$\mathfrak{M}, w \vDash \neg\alpha$	$\Leftrightarrow$	$\mathfrak{M}, w \nvDash \alpha$
$\mathfrak{M}, w \vDash \alpha \wedge \beta$	$\Leftrightarrow$	$\mathfrak{M}, w \vDash \alpha$ and $\mathfrak{M}, w \vDash \beta$
$\mathfrak{M}, w \vDash \alpha \vee \beta$	$\Leftrightarrow$	$\mathfrak{M}, w \vDash \alpha$ or $\mathfrak{M}, w \vDash \beta$
$\mathfrak{M}, w \vDash \Diamonddot\phi$	$\Leftrightarrow$	for some $X \in N(w)$: $\mathfrak{M}, X \Vdash \phi$
$\mathfrak{M}, X \Vdash \phi \wedge \psi$	$\Leftrightarrow$	$\mathfrak{M}, X \Vdash \phi$ and $\mathfrak{M}, X \Vdash \psi$
$\mathfrak{M}, X \Vdash \phi \vee \psi$	$\Leftrightarrow$	$\mathfrak{M}, X \Vdash \phi$ or $\mathfrak{M}, X \Vdash \psi$
$\mathfrak{M}, X \Vdash \Diamondtimes\alpha$	$\Leftrightarrow$	for some $v \in X$: $\mathfrak{M}, v \vDash \alpha$

The following hold:

$$\vDash \boxdot\, \phi \leftrightarrow \neg \Diamonddot \neg\phi \qquad \Vdash \boxtimes\, \alpha \leftrightarrow \neg \Diamondtimes \neg\alpha$$

$$\vDash \boxdot\, (\phi \wedge \psi) \leftrightarrow \boxdot\phi \wedge \boxdot\psi \qquad \Vdash \boxtimes\, (\alpha \wedge \beta) \leftrightarrow \boxtimes\alpha \wedge \boxtimes\beta$$

$$\vDash \Diamonddot\, (\phi \vee \psi) \leftrightarrow \Diamonddot\phi \vee \Diamonddot\psi \qquad \Vdash \Diamondtimes\, (\alpha \vee \beta) \leftrightarrow \Diamondtimes\alpha \vee \Diamondtimes\beta$$

$$\frac{\Vdash \phi \leftrightarrow \psi}{\vDash \boxdot\,\phi \leftrightarrow \boxdot\psi} \qquad \frac{\Vdash \phi \leftrightarrow \psi}{\vDash \Diamonddot\,\phi \leftrightarrow \Diamonddot\psi} \qquad \frac{\vDash \alpha \leftrightarrow \beta}{\Vdash \boxtimes\,\alpha \leftrightarrow \boxtimes\beta} \qquad \frac{\vDash \alpha \leftrightarrow \beta}{\Vdash \Diamondtimes\,\alpha \leftrightarrow \Diamondtimes\beta}$$

Normal form

Lemma (Normal form)

Each point-formula $\gamma \in \mathcal{L}^{\mathrm{ps}}(\boxdot, \boxtimes)$ *is equivalent to a Boolean combination of point-formulas with the INL-shape* $\Diamond\!\!\!\cdot\,(\otimes\alpha_1 \wedge \cdots \wedge \otimes\alpha_n \wedge \boxtimes\beta)$ *with the same propositional letters.*

Just need to turn each $\Diamond\!\!\!\cdot\,\phi$ into a boolean combination of INL-shaped formulas:

- turn ϕ into a disjunction normal form of $\psi^1 \vee \cdots \vee \psi^n$ ($n > 0$) s.t. each ψ^i is a conjunction of some $\boxtimes\beta$ and $\otimes\alpha$.
- turn $\Diamond\!\!\!\cdot\,(\psi^1 \vee \cdots \vee \psi^n)$ into the equivalent $\Diamond\!\!\!\cdot\,\psi^1 \vee \cdots \vee \Diamond\!\!\!\cdot\,\psi^n$.
- each $\Diamond\!\!\!\cdot\,\psi^i$ is $\Diamond\!\!\!\cdot\,(\otimes\alpha_1 \wedge \cdots \wedge \otimes\alpha_m \wedge \boxtimes\beta_1 \wedge \cdots \wedge \boxtimes\beta_k)$ which is equivalent to its INL-shape:

$$\Diamond\!\!\!\cdot\,(\otimes\alpha_1 \wedge \cdots \wedge \otimes\alpha_m \wedge \boxtimes(\beta_1 \wedge \cdots \wedge \beta_k))$$

Expressivity

Each $\mathcal{L}^{\mathtt{inl}}(\Box)$-formula can obviously be rewritten to a point-formula, of $\mathcal{L}^{\mathtt{ps}}(\boxdot, \boxtimes)$ thus:

Theorem

$\mathcal{L}^{\mathtt{ps}}(\boxdot, \boxtimes)$ *and* $\mathcal{L}^{\mathtt{inl}}(\Box)$ *are equally expressive.*

We can define back and forth translations between $\mathcal{L}^{\mathtt{inl}}(\Box)$ and the point-formulas of $\mathcal{L}^{\mathtt{ps}}(\boxdot, \boxtimes)$. This allows us to transfer results of PSNL to INL.

Hilbert system and sequent calculus

Two-sorted Hilbert system $HK^{\boxdot}_{\boxtimes}$

Axioms

TAUTP	$\vdash_p CPL_p$
DIST$\boxdot$	$\vdash_p \boxdot(\phi \to \psi) \to (\boxdot\phi \to \boxdot\psi)$
TAUTS	$\vdash_s CPL_s$
DIST$\boxtimes$	$\vdash_s \boxtimes(\alpha \to \beta) \to (\boxtimes\alpha \to \boxtimes\beta)$

CPL_p/CPL_s stands for classical tautologies for point-/set-formulas.

Rules

MPP $\dfrac{\vdash_p \alpha \quad \vdash_p \alpha \to \beta}{\vdash_p \beta}$

MPS $\dfrac{\vdash_s \phi \quad \vdash_s \phi \to \psi}{\vdash_s \psi}$

NEC$\boxdot$ $\dfrac{\vdash_s \phi}{\vdash_p \boxdot \phi}$

NEC$\boxtimes$ $\dfrac{\vdash_p \alpha}{\vdash_s \boxtimes \alpha}$

A proof of $\vdash_p \alpha$ ($\vdash_s \phi$) is a finite sequence of both $\vdash_p$ and $\vdash_s$ statements ending with $\vdash_p \alpha$ ($\vdash_s \phi$).

For any set of point-formulas $\Gamma \cup \{\alpha\}$, we write $\Gamma \vdash_p \alpha$ iff there are finitely many $\beta_1, \ldots, \beta_n \in \Gamma$ s.t. $\vdash_p \beta_1 \wedge \cdots \wedge \beta_n \to \alpha$ is provable. Similarly for set-formulas $\Sigma \vdash_s \phi$.

Recall: Hilbert system for INL

Crucial axioms:

- $\Box(\alpha_1, \ldots, \alpha_j; \alpha_0) \to \Box(\alpha_1, \ldots, \alpha_j; \alpha_0 \vee \eta)$
- $\Box(\alpha_1, \ldots, \alpha_j, \phi; \alpha_0) \to \Box(\alpha_1, \ldots, \alpha_j, \phi \vee \psi; \alpha_0)$
- $\Box(\alpha_1, \alpha_2, \ldots, \alpha_j; \alpha_0) \to \Box(\alpha_2, \ldots, \alpha_j; \alpha_0)$
- $\Box(\alpha_1, \ldots, \alpha_j; \alpha_0) \to \Box(\alpha_1, \ldots, \alpha_j, \alpha_i; \alpha_0)$ where $i \in \{1, \ldots, j\}$
- $\Box(\alpha_1, \ldots, \alpha_j, \eta; \alpha_0) \to \Box(\alpha_1, \ldots, \alpha_j, \eta \wedge \alpha_0; \alpha_0)$
- $\neg \Box (\alpha_1, \ldots, \alpha_j, \bot; \alpha_0)$
- $\Box(\alpha_1, \ldots, \alpha_n; \beta) \to (\Box(\alpha_1, \ldots, \alpha_n, \gamma; \beta) \vee \Box(\alpha_1, \ldots, \alpha_n; \beta \wedge \neg\gamma))$

`Case` schema:

$$\vdash_{\mathtt{p}} \diamonddot(\circledast\alpha_1 \wedge \cdots \wedge \circledast\alpha_n \wedge \boxtimes\beta) \to \diamonddot(\circledast\alpha_1 \wedge \cdots \wedge \circledast\alpha_n \wedge \circledast\gamma \wedge \boxtimes\beta) \vee \diamonddot(\circledast\alpha_1 \wedge \cdots \wedge \circledast\alpha_n \wedge \boxtimes(\beta \wedge \neg\gamma))$$

Sample derivation

We only prove the following simplified ($n = 1$) schema:

$$\vdash_{\mathbf{p}} \diamonddot(\diamondtimes\alpha \wedge \boxtimes\beta) \rightarrow \diamonddot(\diamondtimes\alpha \wedge \diamondtimes\gamma \wedge \boxtimes\beta) \vee \diamonddot(\diamondtimes\alpha \wedge \boxtimes(\beta \wedge \neg\gamma))$$

Proof.

By tautologies for set formulas and normality of $\boxtimes$:

$$\vdash_{\mathbf{s}} \diamondtimes\alpha \wedge \boxtimes\beta \rightarrow (\diamondtimes\alpha \wedge \diamondtimes\gamma \wedge \boxtimes\beta) \vee (\diamondtimes\alpha \wedge \neg \diamondtimes \gamma \wedge \boxtimes\beta),$$

$$\vdash_{\mathbf{s}} \diamondtimes\alpha \wedge \boxtimes\beta \rightarrow (\diamondtimes\alpha \wedge \diamondtimes\gamma \wedge \boxtimes\beta) \vee (\diamondtimes\alpha \wedge \boxtimes(\neg\gamma \wedge \beta)).$$

By the admissible monotonicity rule,

$$\vdash_{\mathbf{p}} \diamonddot(\diamondtimes\alpha \wedge \boxtimes\beta) \rightarrow \diamonddot((\diamondtimes\alpha \wedge \diamondtimes\gamma \wedge \boxtimes\beta) \vee (\diamondtimes\alpha \wedge \boxtimes(\beta \wedge \neg\gamma))),$$

and then:

$$\vdash_{\mathbf{p}} \diamonddot(\diamondtimes\alpha \wedge \boxtimes\beta) \rightarrow \diamonddot(\diamondtimes\alpha \wedge \diamondtimes\gamma \wedge \boxtimes\beta) \vee \diamonddot(\diamondtimes\alpha \wedge \boxtimes(\beta \wedge \neg\gamma)) \qquad \square$$

Strong completeness

To show $\Gamma \vdash_{\mathbf{p}} \alpha$ iff $\Gamma \vDash \alpha$ and $\Sigma \vdash_{\mathbf{s}} \phi$ iff $\Sigma \Vdash \phi$.

Completeness via canonical model.

To show each $\vdash_{\mathbf{p}}$-consistent set of $\mathcal{L}^p$ has a pointed model, and each $\vdash_{\mathbf{s}}$-consistent set of $\mathcal{L}^s$ has a model w.r.t. a set X. We build a single canonical model $\mathfrak{M}^c = \langle W^c, N^c, V^c \rangle$ where

- W^c is the set of $\vdash_{\mathbf{p}}$-MCSs of point-formulas,
- $N^c(\Delta) = \{X \subseteq W^c \mid \Delta^\flat \subseteq \text{supp}(X)\}$ for each $\Delta \in W^c$,
- $V^c(p) = \{\Delta \in W^c \mid p \in \Delta\}$;

where $\Delta^\flat := \{\phi \mid \boxdot\phi \in \Delta\}$ and supp(X) is the collection of all set-formulas *supported* by $X \subseteq W^c$ in the following sense:

X supports $\boxtimes\alpha$	iff	$\alpha \in \Theta$ for all $\Theta \in X$
X supports $\neg\phi$	iff	X does not support ϕ
X supports $\phi \to \psi$	iff	X does not support ϕ or X supports ψ.

Strong completeness

Claim (#) Every $\vdash_{\mathbf{s}}$-consistent set of set-formulas is supported by an $X \subseteq W^c$.

Lemma (Truth lemma)

For every point-formula α, set-formula ϕ, $\Delta \in W^c$ and $X \subseteq W^c$:

$$\alpha \in \Delta \text{ iff } \mathfrak{M}^c, \Delta \vDash \alpha \quad \textit{and} \quad \phi \in supp(X) \text{ iff } \mathfrak{M}^c, X \Vdash \phi$$

In an induction, Boolean cases are trivial, and here we show:

1. $\boxdot\phi \in \Delta$ iff $\mathfrak{M}^c, \Delta \vDash \boxdot\phi$
2. $\boxtimes\alpha \in \text{supp}(X)$ iff $\mathfrak{M}^c, X \Vdash \boxtimes\alpha$

Claim # is needed in (1) $\Rightarrow$.

The canonical model construction can be transformed into an equivalent one in the setting of INL.

Sequent calculus G3psnl

A 2-sorted version of G3k for K features two sorts of sequents, $\Rightarrow$ and $\Rrightarrow$ for point-and set-formulas.

$$\frac{}{p, \Gamma \Rightarrow \Delta, p}\,(\mathtt{pAx}) \qquad \frac{}{\bot, \Gamma \Rightarrow \Delta}\,(\mathtt{pL}\bot)$$

$$\frac{\Pi \Rrightarrow \Sigma, \phi}{\neg\phi, \Pi \Rrightarrow \Sigma}\,(\mathtt{sL}\neg) \qquad \frac{\phi, \Pi \Rrightarrow \Sigma}{\Pi \Rrightarrow \Sigma, \neg\phi}\,(\mathtt{sR}\neg)$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \to \beta, \Gamma \Rightarrow \Delta}\,(\mathtt{pL}\to) \qquad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \to \beta}\,(\mathtt{pR}\to)$$

$$\frac{\Pi \Rrightarrow \Sigma, \phi \quad \psi, \Pi \Rrightarrow \Sigma}{\phi \to \psi, \Pi \Rrightarrow \Sigma}\,(\mathtt{sL}\to) \qquad \frac{\phi, \Pi \Rrightarrow \Sigma, \psi}{\Pi \Rrightarrow \Sigma, \phi \to \psi}\,(\mathtt{sR}\to)$$

$$\frac{\Pi \Rrightarrow \sigma}{\boxdot\Pi, \Gamma \Rightarrow \Delta, \boxdot\sigma}\,(\mathtt{p}\Box) \qquad \frac{\Gamma \Rightarrow \delta}{\boxtimes\Gamma, \Pi \Rrightarrow \Sigma, \boxtimes\delta}\,(\mathtt{s}\Box)$$

G3psnl is sound and complete, and also admits Cut and support mechanical proof search.

Constructing uniform interpolats

Uniform Interpolation Property (UIP)

- Let Q be a finite set of prop. var.'s.
- A pre-interpolant of $\langle \beta, Q \rangle$ is a formula θ that meets:
  - $\mathcal{V}(\theta) \subseteq \mathcal{V}(\beta) \setminus Q$;
  - $\Vdash \theta \to \beta$;
  - for each α, if $\mathcal{V}(\alpha) \cap Q = \varnothing$ and $\Vdash \alpha \to \beta$, then $\Vdash \alpha \to \theta$.
- A post-interpolant of $\langle \beta, Q \rangle$ is a formula θ that meets:
  - $\mathcal{V}(\theta) \subseteq \mathcal{V}(\beta) \setminus Q$;
  - $\Vdash \beta \to \theta$;
  - for each α, if $\mathcal{V}(\alpha) \cap Q = \varnothing$ and $\Vdash \beta \to \alpha$, then $\Vdash \theta \to \alpha$.
- Uniform interpolation property UIP:
  For each β and Q, pre- and post-interpolant exist.

Uniform Interpolation Property (UIP)

In a logic with classical ¬, it is sufficient to ensure existence of either pre- or post-interpolant:

- if θ is a pre-interpolant of $\langle \neg\alpha, Q \rangle$, then $\neg\theta$ is a post-interpolant of $\langle \alpha, Q \rangle$.

Pre-(post-)interpolant is unique modulo equivalence:

- Interpolants trigger clause (iii) of each other.

Due to [Pitts 1992], there is a method to establish UIP of a logic via a sequent calculus that supports proof-search; [Bílková 2007] extends that method to many modal logics.

Using Pitts-Bílková's method to show UIP of PSNL (and INL).

An adequate notion of UIP for PSNL

Let Q be a finite set of propositional letters. Since PSNL has classical negations (for both sorts of formulas), it is sufficient to find only pre-interpolants (for both sorts).

For $\beta \in \mathcal{F}_p$, a pre-interpolant of $\langle \beta, Q \rangle$ is a formula $\theta \in \mathcal{F}_p$ s.t.:

- $\mathcal{V}(\theta) \subseteq \mathcal{V}(\beta) \setminus Q$;
- $\vDash \theta \to \beta$;
- for each $\alpha \in \mathcal{F}_p$, if $\mathcal{V}(\alpha) \cap Q = \varnothing$ and $\vDash \alpha \to \beta$, then $\vDash \alpha \to \theta$.

For $\psi \in \mathcal{F}_s$, a pre-interpolant of $\langle \psi, Q \rangle$ is a formula $\xi \in \mathcal{F}_s$ s.t.:

- $\mathcal{V}(\xi) \subseteq \mathcal{V}(\psi) \setminus Q$;
- $\Vdash \xi \to \psi$;
- for each $\phi \in \mathcal{F}_s$, if $\mathcal{V}(\phi) \cap Q = \varnothing$ and $\Vdash \phi \to \psi$, then $\Vdash \xi \to \psi$.

UIP: For each $\beta \in \mathcal{F}_p$, $\psi \in \mathcal{F}_s$ and Q, pre-interpolants exist.

Applying Pitts-Bílková's method to PSNL

It is sufficient to verify the sequent version of PSNL's UIP.

Given Q is a finite set of propositional letters, to show:

1. For each point-sequent $\Gamma \Rightarrow \Delta$, there is $\theta^Q_{\Gamma\Delta} \in \mathcal{F}_p$ s.t.:
   - $\mathcal{V}(\theta^Q_{\Gamma\Delta}) \subseteq \mathcal{V}(\Gamma, \Delta) \setminus Q$;
   - $\vdash \Gamma, \theta^Q_{\Gamma\Delta} \Rightarrow \Delta$;
   - $\vdash \Omega \Rightarrow \theta^Q_{\Gamma\Delta}, \Upsilon$ for every point-sequent $\Omega \Rightarrow \Upsilon$ s.t. $\mathcal{V}(\Omega, \Upsilon) \cap Q = \varnothing$ and $\vdash \Omega, \Gamma \Rightarrow \Delta, \Upsilon$.
2. For each set-sequent $\Pi \Rrightarrow \Sigma$, there is $\xi^Q_{\Pi\Sigma} \in \mathcal{F}_s$ s.t.:
   - $\mathcal{V}(\xi^Q_{\Pi\Sigma}) \subseteq \mathcal{V}(\Pi, \Sigma) \setminus Q$;
   - $\vdash \Pi, \xi^Q_{\Pi\Sigma} \Rrightarrow \Sigma$;
   - $\vdash \Phi \Rrightarrow \xi^Q_{\Pi\Sigma}, \Psi$ for every set-sequent $\Phi \Rrightarrow \Psi$ s.t. $\mathcal{V}(\Phi, \Psi) \cap Q = \varnothing$ and $\vdash \Phi, \Pi \Rrightarrow \Sigma, \Psi$.

Applying Pitts-Bílková's method to PSNL

Two-sorted extension of Bílková's construction for K.

- For a non-empty point-/set-sequent s:
  - $\mathrm{c}(s)$:= the closure of s under inverted Boolean schemes; $\mathrm{c}(s)$ is always finite, and share the same sort with s.
  - s is said to be critical, if s is non-empty and on it no inverted Boolean rule scheme is applicable.
  - let $\mathrm{cl}(s) := \{x \in \mathrm{c}(s) \mid x \text{ is critical}\}$.
- For a multi-set $\Theta \subseteq \mathcal{F}_p$, let
  - $\Theta^0 := \{\theta \in \Theta \mid \theta \text{ is prime}\}$;
  - $\Theta^\natural := \{\theta \in \Theta \mid \theta \text{ is } \boxdot\text{-prefixed}\}$;
  - $\Theta^\flat := \{\xi \mid \boxdot\xi \in \Theta^\natural\}$.
- Likewise, for a multi-set $\Xi \subseteq \mathcal{F}_s$, let
  - $\Xi^\natural := \{\xi \in \Xi \mid \xi \text{ is } \boxtimes\text{-prefixed}\}$;
  - $\Xi^\flat := \{\theta \mid \boxtimes\theta \in \Xi^\natural\}$.

Applying Pitts-Bílková's method to PSNL

Construct $\theta^Q_{\Gamma\Delta}$ and $\xi^Q_{\Pi\Sigma}$ by a mutual induction on sequents:

$$\theta^Q_{\Gamma\Delta} := \begin{cases} \bot & \text{if } \Gamma = \Delta = \varnothing \\ \top & \text{else if } \Gamma \Rightarrow \Delta \text{ is critical and } Q \cap \Gamma^0 \cap \Delta^0 \neq \varnothing \\ \multicolumn{2}{l}{\diamonddot \xi^Q_{\Gamma^\flat \varnothing} \vee \bigvee_{\sigma \in \Delta^\flat} \boxdot \xi^Q_{\Gamma^\flat\{\sigma\}} \vee \bigvee \neg(\Gamma^0 \smallsetminus Q) \vee \bigvee(\Delta^0 \smallsetminus Q)} \\ \multicolumn{2}{l}{\quad \text{else if } \Gamma \Rightarrow \Delta \text{ is critical and } Q \cap \Gamma^0 \cap \Delta^0 = \varnothing} \\ \bigwedge_{i \in I} \theta^Q_{\Gamma_i \Delta_i} & \text{else, where } \mathrm{cl}(\Gamma \Rightarrow \Delta) = \{\Gamma_i \Rightarrow \Delta_i\}_{i \in I} \end{cases}$$

$$\xi^Q_{\Pi\Sigma} := \begin{cases} \otimes\bot & \text{if } \Pi = \Sigma = \varnothing \\ \otimes \theta^Q_{\Pi^\flat \varnothing} \vee \bigvee_{\delta \in \Sigma^\flat} \boxtimes \theta^Q_{\Pi^\flat\{\delta\}} & \text{else if } \Pi \Rrightarrow \Sigma \text{ is critical} \\ \bigwedge_{i \in I} \xi^Q_{\Pi_i \Sigma_i} & \text{else, where } \mathrm{cl}(\Pi \Rrightarrow \Sigma) = \{\Pi_i \Rrightarrow \Sigma_i\}_{i \in I} \end{cases}$$

UIP for PSNL and INL

Theorem

Pre- and post-interpolants can be constructed given a finite set of propositional letters and any formula of $\mathcal{L}^{\mathrm{ps}}(\boxdot, \boxtimes)$.

UIP of INL also follows by translation.

Note that it is unclear how to apply Pitts-Bílková's method directly on the sequent calculus G3inl for INL.

It was suggested UIP of PSNL and INL can be proved semantically by coalgebraic method via definability of bisimulation quantifiers.

Conclusions and future work

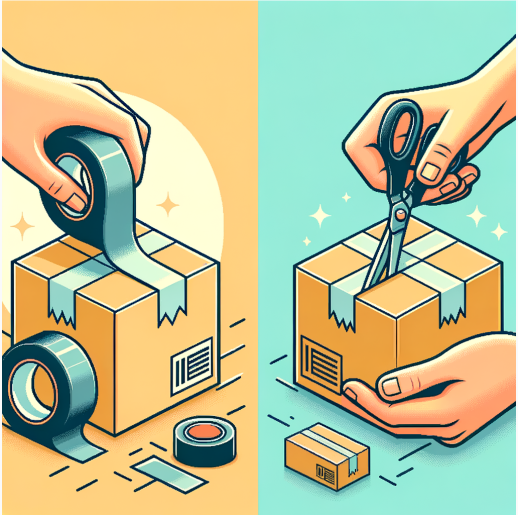
Conclusions: Making simple things simple!

- Breaking the INL-bundles simplifies the techniques
- PSNL is intuitive to use
- It does not increase expressivity
- Multi-sorted language makes use of “non-formulas” in INL
- Bridging rules connects different subsystems
- UIP can be shown constructively

Applications to other bundle-based language: social-friendly coalition logic, Aristotelian modal logic …

Other connections to be explored: team semantics, coalgebraic modal logic…

Bundling or unbundling? That is the question.



The answer: it depends!

Thanks for your attention!