A Bundled Approach to Deontic Logic

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Background and Motivation

Phenomena in natural language as “Icebergs”

What shall we do when seeing an iceberg?

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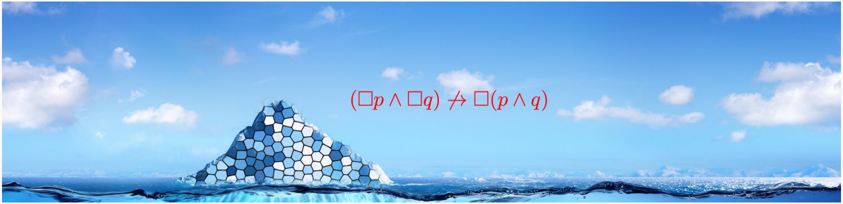
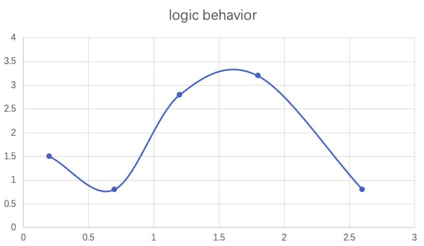


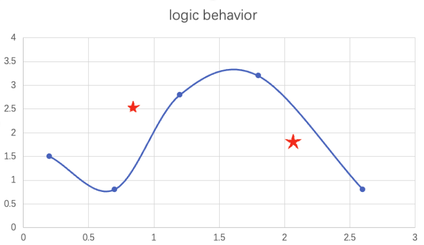
Using various logic techniques to fit the "data"...



Phenomena in natural language as "Icebergs"

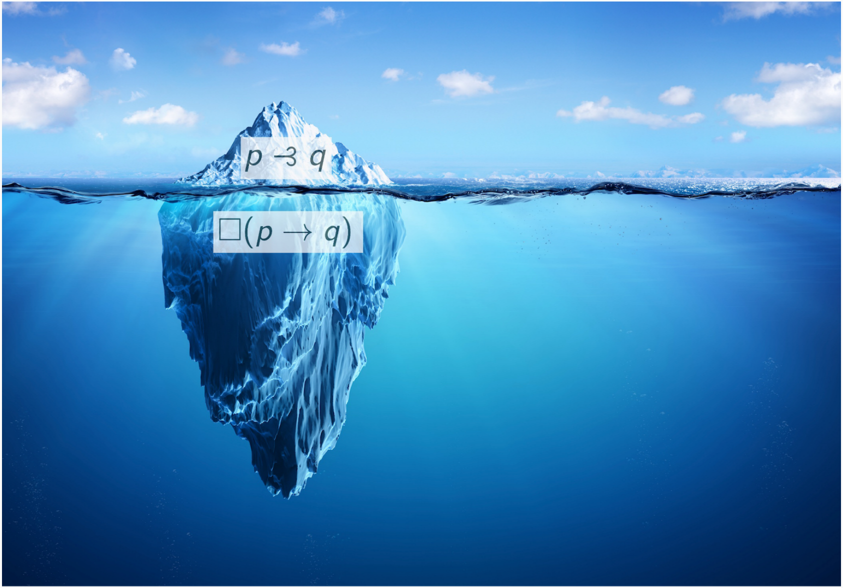
New phenomena by looking at the iceberg from other angles ...

Using logic techniques to fit the new "data"...



Diving to understanding why

Implicit vs. Explicit



For harder cases, special diving equipment is needed

Our bundled approach (based on [Wang and Wang, 2023])

Core Idea: Many innocent-looking modalities are actually more complicated constructions with inner structures – I call them *bundles*.

The bundles of a quantifier and modality packed together (e.g., $\exists x\Box, \forall x\Box$) play important roles in the epistemic logics of know-wh.

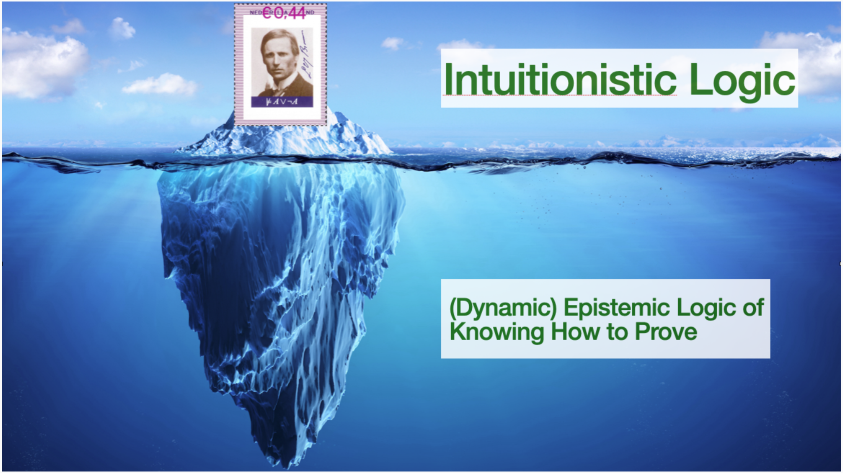
For example, knowing how to achieve φ iff there exists a way x such that it is known x can make sure φ ($\exists\Box$).

Check `wangyanjing.com/pubtype/bkt/` for recent papers on the logics of *knowing what/how/why ...* .

It also leads to:

- New Decidable fragments of first-order modal logic (e.g., [Liu et al., 2023]).
- Epistemic interpretations of non-classical logics (last time at AC+LIRa seminar) e.g., [Wang et al., 2022].

For harder cases, special diving equipment is needed



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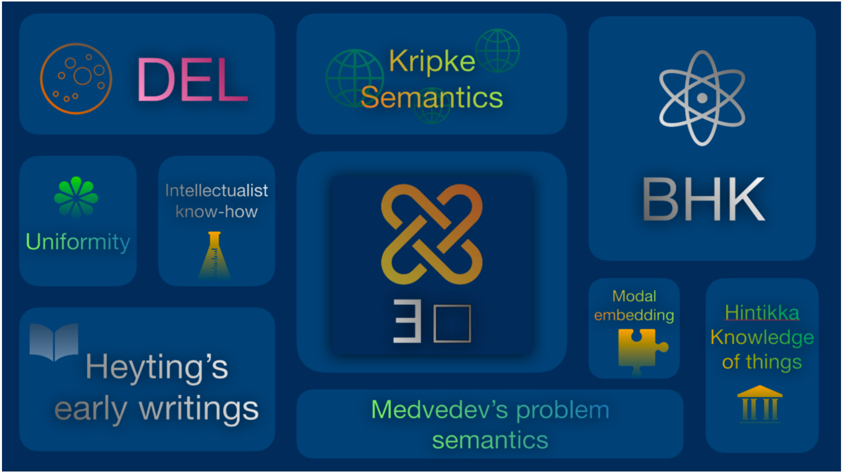

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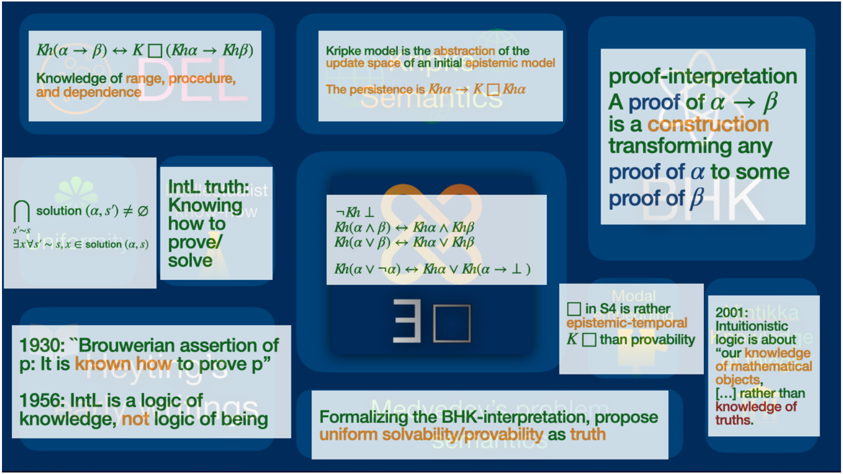

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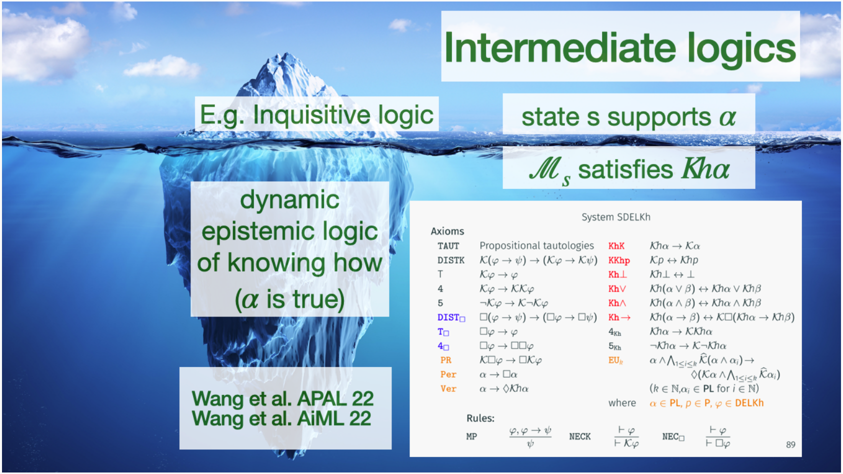

Deontic Logic/Modality: another sea with lots of icebergs...

There are many logical puzzles in Standard Deontic Logic (SDL), deviating from normal modal logic when taking Obligation ($\mathbf{O}$) as a $\Box$ and Permission ($\mathbf{P}$) as a $\Diamond$.

Among many others:

- Ross' paradox: $\mathbf{O}p \rightarrow \mathbf{O}(p \vee q)$ and $\mathbf{P}p \rightarrow \mathbf{P}(p \vee q)$ are intuitively invalid, but valid in SDL.
- Free choice: $\mathbf{P}(p \vee q) \rightarrow \mathbf{P}p \wedge \mathbf{P}q$ is intuitively valid, but logically invalid in SDL.

Free choice is an intriguing linguistic phenomenon in general.

In this talk, we focus on *Strong* Permission ($\mathbf{P}$), the permissions *explicitly granted* rather than simply not being forbidden. Strong permissions exhibit the property of free choice (FCP).

Our basic observations

- Deontic modalities may be more than what they appear to be!
    - Are they also *bundles* of a quantifier and a usual modality?
    - but, quantifying over what?
- Formulas inside deontic modalities might *not* be propositions
    - Then what are they?
    - How can we treat them formally?

Further observations regarding quantifiers and bundles

If a hidden quantifier were present, what would it quantify over?

- The distinction between action types and tokens (well-known)
- Deontic sentences are about action types
- But the semantics may be about tokens of those types

What could be the bundle for strong permission **P**?

- It is clearly not $\exists x \Box$, but it might be $\forall x \Diamond$ (Hintikka 1971).
- $\mathbf{P}\alpha$: each token of action type α is executable on some deontically ideal world.
- Free choice for permission: $\mathbf{P}(\alpha \vee \beta) \rightarrow \mathbf{P}\alpha \wedge \mathbf{P}\beta$
- You are permitted to take one day off next week. All the relevant token should be executable.

Further observations: formalizing action type and token

- Propositional formulas as action types
- They do not have truth values *per se*, though can be assigned!
- They can be viewed as collections of action tokens
- We borrow the BHK-like formalism to capture them

Brouwer-Heyting-Kolmogorov (BHK) interpretation

BHK *proof* interpretation of *connectives*:

(H1) A proof of $\alpha \wedge \beta$ is given by presenting a proof of α and a proof of β

(H2) A proof of $\alpha \vee \beta$ is given by presenting either a proof of α or a proof of β

(H3) A proof of $\alpha \to \beta$ is a construction which *transforms* any proof of α into some proof of β

(H4) Absurdity $\perp$ has *no* proof.

$\neg\alpha$ is the abbriviation of $\alpha \to \perp$.

Further observations: formalizing action type and token

- Propositional formulas as action types
- They do not have truth values *per se*, though can be assigned!
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	Intuitionistic Logic	Deontic Logic
prop. formulas	type of problems	type of actions
token	solution/proof	individual act
modality	know-how	permission
bundle	$\exists\Box$	$\forall\Diamond$

Clarification

Two readings of "*You may take coffee or tea*".

- performative
- descriptive

Our work is more on the descriptive side. Wait till the end for nested permissions, which is connected to the performative perspective.

Preview: beyond fitting the known data

Valid in our framework			
FC	$\mathbf{P}(\alpha \vee \beta) \leftrightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$	CD	$\mathbf{P}(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow \mathbf{P}((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$
CE	$\mathbf{P}(\alpha \wedge \beta) \rightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$	DCl	$\mathbf{P}((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \rightarrow \mathbf{P}(\alpha \vee (\beta \wedge \gamma))$
Invalid in our framework			
CA	$(\mathbf{P}\alpha \wedge \mathbf{P}\beta) \rightarrow \mathbf{P}(\alpha \wedge \beta)$	DCr	$\mathbf{P}(\alpha \vee (\beta \wedge \gamma)) \rightarrow \mathbf{P}((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$
RP	$\mathbf{P}\alpha \rightarrow \mathbf{P}(\alpha \vee \beta)$	EX	$\mathbf{P}\alpha \rightarrow \mathbf{P}(\alpha \wedge \alpha)$

DCr is invalid: imagine you are given a coupon that permits you to take a hamburger *or* a menu of French fries *and* salad, this does not mean you can take a hamburger *or* fries, *and* a hamburger *or* salad.

CA and DCr are valid in Boolean-algebra-based approaches, such as [Castro and Kulicki, 2014, Trypuz and Kulicki, 2015]; CD is invalid in [Bentzen, 2014]; DCr is valid in the hybrid approach based on BSML [Aloni, 2022]; and CE is not valid in [van Benthem, 1979].

About the "innocent" EX: $\mathbf{P}\alpha \to \mathbf{P}(\alpha \wedge \alpha)$

It is not as innocent as it may seem. Under free choice and acceptable distribution, it leads to the unacceptable $\mathbf{P}(\alpha \vee \beta) \to \mathbf{P}(\alpha \wedge \beta)$!

$$
\begin{array}{lll}
 & \mathbf{P}(\alpha \vee \beta) & \\
\Longrightarrow & \mathbf{P}((\alpha \vee \beta) \wedge (\alpha \vee \beta)) & (\mathtt{EX}) \\
\Longleftrightarrow & \mathbf{P}(((\alpha \vee \beta) \wedge \alpha) \vee ((\alpha \vee \beta) \wedge \beta)) & (\mathtt{CD}) \\
\Longleftrightarrow & \mathbf{P}((\alpha \vee \beta) \wedge \alpha) \wedge \mathbf{P}((\alpha \vee \beta) \wedge \beta) & (\mathtt{FC}) \\
\Longleftrightarrow & \mathbf{P}((\alpha \wedge \alpha) \vee (\alpha \wedge \beta)) \wedge \mathbf{P}((\beta \wedge \alpha) \vee (\beta \wedge \beta)) & (\mathtt{CD}, \text{commutativity}) \\
\Longleftrightarrow & \mathbf{P}(\alpha \wedge \alpha) \wedge \mathbf{P}(\alpha \wedge \beta) \wedge \mathbf{P}(\beta \wedge \alpha) \wedge \mathbf{P}(\beta \wedge \beta) & (\mathtt{FC}) \\
\Longrightarrow & \mathbf{P}(\alpha \wedge \beta) & (\mathtt{TAUT})
\end{array}
$$

In a lottery, you are allowed to pick a ticket does not mean you are allowed to pick one and yet another one. The invalidity of EX signals some resource-bounded flavor as in (deontic) linear logic (Lokhorst 1997, Backer 2010).

Language and Semantics

Language AT of action types

Definition (Action Type AT)

Given a countable set of propositional letters P, the language of action types (**AT**) is defined as follows:

$$\alpha := p \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha)$$

where $p \in P$.

We use atomic propositional letters to represent atomic action types like "drink coffee", "do homework", "go to hospital", etc. Complex action types like 'eat cookies and drink coffee", "do homework or play computer games" can also be expressed.

Language DLSP of Deontic Logic of Strong Permission

Definition (Language DLSP)

Given **AT**, the language of deontic logic for strong permission (**DLSP**) is defined as follows:

$$\varphi := p \mid (\varphi \land \varphi) \mid \neg\varphi \mid \mathbf{P}\alpha$$

where $p \in P$ and $\alpha \in \mathbf{AT}$. Connectives $\vee$ and $\rightarrow$ are defined classically as usual.

We call formulas containing the deontic operator **P** *deontic formulas* and other formulas *non-deontic*.

Action space

Following the BHK-style definition:

Definition (Action Token Space)

Given P and a non-empty set I of atomic action tokens, an action (token) space S is a function based on I and **AT** satisfying the following constraints:

1. $S(p) \neq \emptyset \subseteq I$ for any $p \in P$;
2. $S(\alpha \wedge \beta) = S(\alpha) \times S(\beta)$;
3. $S(\alpha \vee \beta) = S(\alpha) \cup S(\beta)$.

S is a singleton action space if $|S(p)| = 1$ for all $p \in P$. People may treat types and tokens alike.

For example, action tokens for a disjunctive action type $(\alpha \vee \beta)$ are the union of tokens of α and β.

Deontic Model

Definition (Deontic Model)

A deontic model $\mathcal{M}$ for **DLSP** is a tuple (S, W, R, A) where S is an action space, W is a non-empty set of possible worlds, $R \subseteq W \times W$, and A is a binary function over $\mathbf{AT} \times W$ such that for any $p \in P$, $\alpha, \beta \in \mathbf{AT}$ and $w \in W$:

- $A(p, w) \subseteq S(p)$;
- $A(\alpha \wedge \beta, w) = A(\alpha, w) \times A(\beta, w)$;
- $A(\alpha \vee \beta, w) = A(\alpha, w) \cup A(\beta, w)$;

A pointed model is a pair $(\mathcal{M}, w)$ where w is in $\mathcal{M}$. A singleton deontic model is a model based on a singleton action space.

The function A gives each deontially ideal world its *executed* action tokens.

Definition (Semantics)

For any $\varphi \in \mathbf{DLSP}$ and any pointed deontic model $\mathcal{M}, w$ where $\mathcal{M} = (S, W, R, A)$, the satisfaction relation is defined as follows:

$$
\begin{array}{lcl}
\mathcal{M}, w \vDash p & \iff & A(p, w) \neq \emptyset \\
\mathcal{M}, w \vDash (\varphi \wedge \psi) & \iff & \mathcal{M}, w \vDash \varphi \text{ and } \mathcal{M}, w \vDash \psi \\
\mathcal{M}, w \vDash \neg\varphi & \iff & \mathcal{M}, w \nvDash \varphi \\
\mathcal{M}, w \vDash \mathbf{P}\alpha & \iff & \text{for any } a \in S(\alpha), \text{ there is a } v \text{ s.t.} \\
 & & wRv \text{ and } a \in A(\alpha, v)
\end{array}
$$

We use $\vDash_s$ to denote semantic consequence w.r.t. singleton deontic models. We say φ is valid (s-valid) if $\vDash \varphi$ ($\vDash_s \varphi$).

Recall the preview

Valid in our framework			
FC	$\mathbf{P}(\alpha \vee \beta) \leftrightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$	CD	$\mathbf{P}(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow \mathbf{P}((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$
CE	$\mathbf{P}(\alpha \wedge \beta) \to (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$	DCl	$\mathbf{P}((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \to \mathbf{P}(\alpha \vee (\beta \wedge \gamma))$
Invalid in our framework (without further constraints)			
CA	$(\mathbf{P}\alpha \wedge \mathbf{P}\beta) \to \mathbf{P}(\alpha \wedge \beta)$	DCr	$\mathbf{P}(\alpha \vee (\beta \wedge \gamma)) \to \mathbf{P}((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$
RP	$\mathbf{P}\alpha \to \mathbf{P}(\alpha \vee \beta)$	EX	$\mathbf{P}\alpha \to \mathbf{P}(\alpha \wedge \alpha)$

The commutativity and associativity are valid.

$$\mathbf{P}(\alpha \wedge \beta) \leftrightarrow \mathbf{P}(\beta \wedge \alpha) \qquad \mathbf{P}((\alpha \wedge \beta) \wedge \gamma) \leftrightarrow \mathbf{P}(\alpha \wedge (\beta \wedge \gamma))$$
$$\mathbf{P}(\alpha \vee \beta) \leftrightarrow \mathbf{P}(\beta \vee \alpha) \qquad \mathbf{P}((\alpha \vee \beta) \vee \gamma) \leftrightarrow \mathbf{P}(\alpha \vee (\beta \vee \gamma))$$

Invalidity of `DCr`: $\mathbf{P}(\alpha \vee (\beta \wedge \gamma)) \rightarrow \mathbf{P}((\alpha \vee \beta) \wedge (\alpha \vee \gamma))$

The rightmost part below demonstrates the definition of A on u, v, e.g., $A(p, v) = \{a\}$ and $A(q, u) = \{b\}$.

$$S(p) = \{a\}, S(q) = \{b\}, S(r) = \{c\} \qquad \begin{tikzcd} & v & p : \{a\}, r : \emptyset, q : \emptyset \\ w \arrow[ur] \arrow[dr] & & \\ & u & p : \emptyset, q : \{b\}, r : \{c\} \end{tikzcd}$$

$S(p \vee (q \wedge r))$ contains a and (b, c) only, which are executable on v and u respectively, thus $\mathbf{P}(p \vee (q \wedge r))$ is true on w. However, the token (a, c) in $S((p \vee q) \wedge (p \vee r))$ is not executable on u nor v, thus $\mathbf{P}((p \vee q) \wedge (p \vee r))$ is false on w. Note that this model is also a singleton model so `DCr` is not s-valid.

A weaker version of EX over singleton models

$\mathtt{EX} : \mathbf{P}\alpha \to \mathbf{P}(\alpha \wedge \alpha)$ is not valid since every token of α is executable does not mean every pair of α-tokens is executable.

The following formula (denoted by EXP) is valid with respect to the class of singleton deontic models:

$$\vDash_s \mathbf{P}(p_1 \wedge ... \wedge p_k) \to \mathbf{P}(m_1 \cdot p_1 \wedge ... \wedge m_k \cdot p_k),$$

where $p_1, ...p_k \in P$ are pairwise distinct, $k, m_i \in \mathbb{N}_{>0}$ for any $1 \leq i \leq k$. Here $m_i \cdot p_i$ represents the conjunction of m_i copies of p_i.

Proof Systems and Completeness

Proof Systems (no replacement of equals in P)

System $\mathbb{DLSP}$

Axioms	
(TAUT)	Propositional Tautologies
(FC)	$\mathbf{P}(\alpha \vee \beta) \leftrightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$
(CE)	$\mathbf{P}(\alpha \wedge \beta) \rightarrow (\mathbf{P}\alpha \wedge \mathbf{P}\beta)$
($\mathtt{COM}_\wedge$)	$\mathbf{P}(\alpha \wedge \beta) \leftrightarrow \mathbf{P}(\beta \wedge \alpha)$
($\mathtt{ASSO}_\wedge$)	$\mathbf{P}((\alpha \wedge \beta) \wedge \gamma) \leftrightarrow \mathbf{P}(\alpha \wedge (\beta \wedge \gamma))$
(CD)	$\mathbf{P}(\alpha \wedge (\beta \vee \gamma)) \leftrightarrow \mathbf{P}((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))$
Rules	
(MP)	Given φ and $(\varphi \rightarrow \psi)$, infer ψ.

$\mathtt{DC1}$: $\mathbf{P}((\alpha \vee \beta) \wedge (\alpha \vee \gamma)) \rightarrow \mathbf{P}(\alpha \vee (\beta \wedge \gamma))$ is provable.

Normal form

We use $\mathbb{DLSP}$ to rewrite a **DLSP**-formula into a conjunction of formulas in the shape of $\mathbf{P}(p_1 \wedge ... \wedge p_n)$.

$$\mathbf{P}(p_1 \vee (p_2 \wedge ((p_3 \vee p_4) \wedge p_5))).$$

The formula is logically equivalent to

1. $\mathbf{P}p_1 \wedge \mathbf{P}(p_2 \wedge ((p_3 \vee p_4) \wedge p_5))$ $\quad (\mathtt{FC})$
2. $\mathbf{P}p_1 \wedge \mathbf{P}((p_5 \wedge p_2) \wedge (p_3 \vee p_4))$ $\quad (\mathtt{ASSO}_\wedge + \mathtt{COM}_\wedge)$
3. $\mathbf{P}p_1 \wedge \mathbf{P}(((p_5 \wedge p_2) \wedge p_3) \vee ((p_5 \wedge p_2) \wedge p_4))$ $\quad (\mathtt{CD})$
4. $\mathbf{P}p_1 \wedge \mathbf{P}((p_5 \wedge p_2) \wedge p_3) \wedge \mathbf{P}((p_5 \wedge p_2) \wedge p_4)$ $\quad (\mathtt{FC})$

Lemma (Normal Form for $\mathbf{P}\alpha$)

*For any $\alpha \in$ **AT**, $\mathbf{P}\alpha$ is logically equivalent to a formula of the form $(\mathbf{P}\beta_1 \wedge ... \wedge \mathbf{P}\beta_k)$ where for each $1 \leq i \leq k$, β_i is in the shape of $\mathbf{P}(p_1 \wedge ... \wedge p_n)$, which is called a* **normal form** *for $\mathbf{P}\alpha$.*

Normal form

For any formula $\varphi \in \mathbf{DLSP}$, φ is logically equivalent to a formula in the following language (denoted by $\mathbf{DLSP}^*$):

$$\psi ::= p \mid \mathbf{P}(p_1 \wedge ... \wedge p_n) \mid \neg\psi \mid (\psi \wedge \psi),$$

where $p, p_1, ..., p_n \in P$.

To show the completeness, we will construct a model (S, W, R, A) for each consistent set of formulas in the above language.

We need to first build the action space S before constructing the canonical model.

Canonical action space

Now let Σ be a maximally $\mathbb{DLSP}$-consistent set of **DLSP*** formulas.

Definition (Canonical Action Space)

Given Σ, we define S^C_Σ by distinguishing the two cases of $p \in P$:

- If there is an $i \in \mathbb{N}_{>0}$ such that the formula $\neg\mathbf{P}(i \cdot p) \in \Sigma$, assume that n is the least and let $S^C_\Sigma(p) := \{p^1, p^2, ..., p^n\}$, in which each p^i is the propositional letter p superscript with the numeral i.
- If not, i.e., $\mathbf{P}(i \cdot p) \in \Sigma$ for all $i \in \mathbb{N}_{>0}$, let $S^C_\Sigma(p) := \{p^1, p^2, ...\}$.

For any composite $\alpha \in \mathbf{AT}$, we define $S^C_\Sigma(\alpha)$ recursively as in the definition of S.

Note that for distinct $p, q \in P$, $S^C_\Sigma(p) \cap S^C_\Sigma(q) = \emptyset$.

The very idea of canonical model

Based on $S^{\mathcal{C}}_{\Sigma}$, we will build a pointed deontic model $\mathcal{M}^{\mathcal{C}}_{\Sigma}, w$ such that the truth lemma holds. The crucial step is the $\mathbf{P}(p_1 \wedge \ldots \wedge p_n)$ case.

Due to the validity of $\mathtt{COM}_{\wedge}$and $\mathtt{ASSO}_{\wedge}$, we only consider formulas of the form $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k}) \in \Sigma$, where $m_j \cdot p_{t_j}$ means the conjunction of m_j copies of p_{t_j}, and p_{t_i} and p_{t_j} are pairwise distinct.

The idea is simple: given a designated world w, build the accessible worlds according to formulas $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k}) \in \Sigma$.

The subtlety is that we should only realize action tokens that are *necessary* to witness the truth of those φ, but no more, for we also need tokens not realizable to witness $\neg\mathbf{P}(p_1 \wedge \ldots \wedge p_n) \in \Sigma$.

All-distinct action token

These necessary tokens are *all-distinct* action tokens.

Given an action token space, first note that we can treat each action token of type $(p_1 \wedge \ldots \wedge p_n)$ as an n-ary tuple of atomic action tokens modulo paring.

Definition (All-Distinct Token)

An action token of type $(p_1 \wedge \ldots \wedge p_n)$ is *all-distinct* if tokens of the same atomic action type in the tuple are pairwise distinct.

For our canonical model, we will construct a unique successor for each all-distinct token to realize $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k}) \in \Sigma$. But before that, we need to guarantee that such tokens indeed exist.

The existence of all-distinct token

The following lemma shows that the size of the canonical action space is more than enough to guarantee the existence of all-distinct action tokens of the type when $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k}) \in \Sigma$.

Lemma (Existence of All-Distinct Tokens)

For any formula φ *of the form* $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k})$ *where* p_{t_i}, p_{t_j} *are pairwise distinct, if* $\varphi \in \Sigma$*, then for any* $1 \leq j \leq k$*,* $m_j < |S^C_\Sigma(p_{t_j})|$*.*

Proof.

Prove by contradiction. Suppose there is j s.t. $m_j \geq |S^C_\Sigma(p_{t_j})| = n$. Thus by (CE) we have $\varphi \to \mathbf{P}(n \cdot p_{t_j}) \in \Sigma$. Since $\varphi \in \Sigma$ and Σ is maximal, $\mathbf{P}(n \cdot p_{t_j}) \in \Sigma$, which contradicts that Σ is consistent. $\square$

Functional representation for permissions

Fixing an ordering of propositional letters $p_0, p_1, p_2, \ldots$, we only need to consider $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k}) \in \Sigma$ such that p_{t_i} and p_{t_j} are distinct and ordered, e.g., $\mathbf{P}(3 \cdot p_2 \wedge 4 \cdot p_6)$.

Definition

For any φ of the form $\mathbf{P}(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k}) \in \Sigma$ such that p_{t_i} and p_{t_j} are distinct and ordered according to the order of propositional letters, we define $f_\varphi : \mathbb{N} \to \mathbb{N}$ such that

$$f_\varphi(i) = \begin{cases} m_j & i = t_j \text{ for some } 1 \leq j \leq k; \\ 0 & i \neq t_j \text{ for any } 1 \leq j \leq k. \end{cases}$$

For example, $\mathbf{P}(3 \cdot p_2 \wedge 4 \cdot p_6)$ is represented by the function f such that $f(2) = 3$, $f(6) = 4$ and $f(i) = 0$ for any $i \in \mathbb{N} \setminus \{2, 6\}$. We collect these (countably many) functions in F_Σ.

Functional representation for all-distinct tokens

Definition

For any $f \in F_\Sigma$, we define $G_f := \{g : \mathbb{N} \to \mathcal{P}(\bigcup_{p \in P}(S_\Sigma^C(p))) \mid$ for any $i \in \mathbb{N}, g(i) \subseteq S_\Sigma^C(p_i)$ and $|g(i)| = f(i)\}$.

Intuitively, each $g \in G_f$ assigns a subset of the canonical action space of each p_i whose cardinality is $f(i)$. It follows if $f(i) = 0$ then $g(i) = \emptyset$. In fact, each $g \in G_f$ can be treated as an *all-distinct token of the corresponding type to function f*. Let $G_\Sigma = \bigcup\{G_f \mid f \in F_\Sigma\}$.

Proposition

Given a MCS Σ and any distinct $f, f' \in F_\Sigma$, we have: (1) G_f is not empty; (2) $G_f \cap G_{f'} = \emptyset$.

Canonical deontic model

Definition (Canonical Deontic Model)

Given a MCS Σ, we define the model $\mathcal{M}^C_\Sigma = (S^C_\Sigma, W^C, R^C, A^C)$ where:

- $W^C = \{w\} \cup G_\Sigma$; $R^C = \{(w, g) \mid g \in G_\Sigma\}$;
- $A^C(p_i, u) = \begin{cases} S^C_\Sigma(p_i) & \text{if } u = w \text{ and } p_i \in \Sigma, \\ \emptyset & \text{if } u = w \text{ and } p_i \notin \Sigma, \\ u(i) & \text{if } u \in G_\Sigma; \end{cases}$

and $A^C(\alpha, u)$ for composite α is defined as above.

If $g \in G_\Sigma$ then there is a unique $f \in F_\Sigma$ s.t. $g \in G_f$. Intuitively, each $g \in G_f$ realizes some all-distinct token of the formula $\mathbf{P}\alpha \in \Sigma$ corresponding to f, and G_f realize all the necessary tokens.

Lemma (Truth Lemma for $\mathbb{DLSP}$)

Given a MCS Σ. For any $\varphi \in \Sigma$,

$$\mathcal{M}^C_\Sigma, w \vDash \varphi \iff \varphi \in \Sigma.$$

Proof.

Prove by induction on the structure of φ. We only show the inductive case when $\varphi = \mathbf{P}(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k})$.

$\Leftarrow$: Assume that $\varphi \in \Sigma$. We have the corresponding $f_\varphi \in F_\Sigma$. Then each all-distinct token of the type is represented and thus realized by some $g \in G_{f_\varphi}$. And this will guarantee all tokens be realized.

Truth Lemma for $\mathbb{DLSP}$

Proof.

$\Rightarrow$: Assume that $\varphi \notin \Sigma$. To show $\mathcal{M}^C_\Sigma, w \not\models \varphi$, we need to find some token in $S^C_\Sigma(m_1 \cdot p_{t_1} \wedge ... \wedge m_k \cdot p_{t_k})$ cannot be witnessed by any successor. The crucial point here is that our definition of S^C_Σ and A^C together guarantee that some action tokens are indeed **left out** at every $g \in G_\Sigma$. We consider two cases:

- If for any $1 \leq j \leq k$, $m_j \leq |S^C_\Sigma(p_{t_j})|$,
- If there is $1 \leq j \leq k$ such that $m_j > |S^C_\Sigma(p_{t_j})|$,

Proof.

- If for any $1 \leq j \leq k$, $m_j \leq |S^C_\Sigma(p_{t_j})|$, we take an all-distinct token $x \in S(m_1 \cdot p_{t_1} \wedge \ldots \wedge m_k \cdot p_{t_k})$ and show it is not realizable in G_Σ, thus $\mathcal{M}^C_\Sigma, w \not\models \varphi$. Suppose not, so there is a $g \in G_\Sigma$ that realizes x, then there is a unique f such that $g \in G_f$. Since g realizes all-distinct token x, then we have $f(t_j) = |g(t_j)| = |A^C(p_{t_j}, g)| \geq m_j$ for any $1 \leq j \leq k$. Due to our construction, there must be a $\chi \in \Sigma$ such that $f = f_\chi$. Therefore, χ must be of the form $\mathbf{P}((m'_1 \cdot p_{t_1} \wedge \ldots \wedge m'_k \cdot p_{t_k}) \wedge (m'_{k+1} \cdot p_{t_{k+1}} \wedge \ldots \wedge m'_{k+l} \cdot p_{t_{k+l}})) \in \Sigma$ such that $m'_j = f(t_j) \geq m_j$. By $(\mathtt{CE})$ and $(\mathtt{MP})$, $\varphi \in \Sigma$, contradicting to the assumption that $\varphi \notin \Sigma$.

Truth Lemma for $\mathbb{DLSP}$

Proof.

- If there is $1 \leq j \leq k$ such that $m_j > |S^C_\Sigma(p_{t_j})|$, thus $S^C_\Sigma(p_{t_j})$ is finite, say $|S^C_\Sigma(p_{t_j})| = n$. Suppose towards a contradiction that $\mathcal{M}^C_\Sigma, w \vDash \psi$. Thus by the validity of $\mathtt{CE}$, $\mathcal{M}^C_\Sigma, w \vDash \mathbf{P}(n \cdot p_{t_j})$. Hence, to realize the token using all the atomic tokens in $S^C_\Sigma(p_{t_j})$, there must be a $g \in G$ such that $A^C(p_{t_j}, g) = g(t_j) = S^C_\Sigma(p_{t_j})$. Further there must be a unique f such that $g \in G_f$ and $f(t_j) = |g(t_j)| = |S^C_\Sigma(p_{t_j})| = n$. Therefore there is a $\chi \in \Sigma$ such that $f = f_\chi$. However this means χ must be in the shape of $\mathbf{P}(n \cdot p_{t_j} \wedge \beta) \in \Sigma$. By ($\mathtt{CE}$), $\mathbf{P}(n \cdot p_{t_j}) \in \Sigma$ contradicting to the fact that $|S^C_\Sigma(p_{t_j})| = n$. Therefore $\mathcal{M}^C_\Sigma, w \nvDash \psi$.

$\square$

Completeness Theorems

Based on the truth lemma, by a Lindenbaum-like argument, we can show:

Theorem (Completeness Theorem for $\mathbb{DLSP}$)

$\mathbb{DLSP}$ *is strongly complete with respect to the class of all deontic models.*

Note that $\mathbb{DLSP}$ is also complete over all *serial* models, i.e., the models where every node has a successor.

System $\mathbb{DLSP}^s$ is the System $\mathbb{DLSP}$ with an extra axiom schema:

$$\boxed{(\mathtt{EXP}) \qquad \mathbf{P}(p_1 \wedge \ldots \wedge p_k) \to \mathbf{P}(m_1 \cdot p_1 \wedge \ldots \wedge m_k \cdot p_k)}$$

Theorem (Completeness Theorem for $\mathbb{DLSP}^s$)

$\mathbb{DLSP}^s$ *is strongly complete with respect to the class of all singleton deontic models.*

Generalizations and extensions

Generalizations

The BHK-definition is just one particular way to define the action types and tokens. We can relax the setting inherited from the proof-interpretation.

For example regarding $\wedge$, we can ask:

- whether token *a* of type α and token *b* of type β on a world necessarily lead to (a, b) of type $(\alpha \wedge \beta)$?
- whether a token of type $(\alpha \wedge \beta)$ necessarily induces the existence of a token of α and the existence of a token of β?
- whether the type $\alpha \wedge \beta$ asks for a token which is in both type α and β?

Essentially we can give different versions of the "semantics" to the connectives. Surprisingly, some changes do not affect the logic. E.g., if we allow the possibility of having token *a* of α and token *b* of type β without having (a, b) of type $(\alpha \wedge \beta)$.

Closure set of tokens

Given an action space S, we define $I_S := \bigcup_{\alpha \in \mathbf{AT}} S(\alpha)$. That is, I_S collects exactly all possible action tokens of all action types under S.

Definition (Closure Set of Tokens)

We say a set $T \subseteq I_S$ of action tokens is *closed* iff

1. If $(a, b) \in T$, then $a \in T$ and $b \in T$;
2. If $(a, b) \in T$, then $(b, a) \in T$;
3. $((a, b), c) \in T$ if and only if $(a, (b, c)) \in T$.

These closure properties validate basic axioms in our system $\mathbb{DLSP}$. However, it is possible for $a, b \in T$ but $(a, b) \notin T$.

This can also model resource-boundedness when $a = b$.

General deontic model

Definition (I-Type General Deontic Model)

A I-type general deontic model $\mathcal{M}^G$ is a 5-ary tuple (S, W, R, A, σ) such that

- (S, W, R, A) is a deontic model.
- $\sigma : W \to \wp(I_S)$ such that for any $w \in W$, $\sigma(w)$ is closed.

We define: $A^G(\alpha, w) := A(\alpha, w) \cap \sigma(w)$.

Definition (II-Type General Deontic Model)

A II-type general deontic model is a 4-ary tuple (S, W, R, σ) such that S, W, R are as usual, and

- $\sigma : W \to \wp(I_S)$ such that for any $w \in W$, $\sigma(w)$ is closed.

We also define function A^G rather by $A^G(\alpha, w) := S(\alpha) \cap \sigma(w)$.

Mutual transformation

Semantics is defined as before except we set $\mathcal{M}, w \Vdash_i \mathbf{P}\alpha \iff$ for any $a \in S(\alpha)$, there is a $v \in W$ such that wRv and $a \in A^G(\alpha, v)$, where $i \in \{\mathrm{I}, \mathrm{II}\}$.

Definition (Disjointed Action Space)

Given a non-empty set I of action tokens, a disjointed action space S is an action space such that for any $p, q \in P$, if $p \neq q$, then $S(p) \cap S(q) = \emptyset$.

Proposition (Mutual Transformation)

*Given a disjointed action space S, for each I-type general deontic model $\mathcal{M} = (S, W, R, A, \sigma)$, there is a II-type general deontic model $\mathcal{M}' = (S, W, R, \sigma')$ such that for any $\varphi \in$ **DLSP** and $w \in W$, $\mathcal{M}, w \Vdash_{\mathrm{I}} \varphi$ if and only if $\mathcal{M}', w \Vdash_{\mathrm{II}} \varphi$; and vice versa.*

Various soundness and completeness results

Deontic Models	A	A: Co	(A, σ)	(A, σ): Co	σ
S	$\mathbb{DLSP}$	$\mathbb{DLSP}$	$\mathbb{DLSP}$	$\mathbb{DLSP}$	$\mathbb{DLSP}$
S: Disjointed	$\mathbb{DLSP}$	$\mathbb{DLSP}$	$\mathbb{DLSP}$	$\mathbb{DLSP}$	$\mathbb{DLSP}$
S: Singleton	$\mathbb{DLSP}^s$	$\mathbb{DLSP}^s$	$\mathbb{DLSP}$	$\mathbb{DLSP}$	$\mathbb{DLSP}$
S: Singleton, Disjointed	$\mathbb{DLSP}^s$	$\mathbb{DLSP}^s$	$\mathbb{DLSP}$	$\mathbb{DLSP}$	$\mathbb{DLSP}$

In the future, we may also relax other closure properties.

Extension: higher-order permissions

Definition (Action Type AT)

Given a countable set of propositional letters P, the language of action types (**AT**) is defined as follows:

$$\alpha := p \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha) \mid \mathbf{P}\alpha$$

Giving a permission itself can also be an action type!

Then we can express $\mathbf{PP}p$, $\neg\mathbf{P}(p \vee \mathbf{P}q)$, $\mathbf{PP}p \to \mathbf{P}p, \mathbf{P}p \wedge \neg\mathbf{PP}p \ldots$

Higher-order permissions matter a lot in security systems, where permissions can be inhierated and transferred.

Action space and tokens

We can give the interpretation for $\mathbf{P}\alpha$ as a type.

- $S(\mathbf{P}\alpha) = \{c_\alpha\}$.
- $A(\mathbf{P}\alpha, w) = \begin{cases} \{c_\alpha\} & w \vDash \mathbf{P}\alpha \\ \emptyset & \text{otherwise} \end{cases}$

Moreover, we still have $w \vDash \alpha$ iff $A(\alpha, w) \neq \emptyset$.

The system $\mathbb{DLSP}$ with the following rule is sound and complete:

$$\boxed{\text{Given } \vdash \bigwedge \overline{\mathbf{P}\alpha} \to \bigwedge \overline{\mathbf{P}\beta}, \text{ infer } \chi \to \chi[\bigwedge \overline{\mathbf{P}\beta} / \bigwedge \overline{\mathbf{P}\alpha}].}$$

With an extra axiom $\mathbf{PP}\alpha \to \mathbf{P}\alpha$, the logic is complete over transitive frames.

Extension: adding negation

Definition (Action Type AT)

$$\alpha := p \mid (\alpha \wedge \alpha) \mid (\alpha \vee \alpha) \mid \mathbf{P}\alpha \mid \neg\alpha$$

$\mathbf{P}\neg\alpha$ intuitively says it is permitted not to do α.

- $S(\neg\alpha) = \{n_\alpha\}$.
- $A(\neg\alpha, w) = \begin{cases} \{n_\alpha\} & A(\alpha, w) = \emptyset \text{ or equivalently } w \not\models \alpha \\ \emptyset & \text{otherwise} \end{cases}$

Given the semantics, we have: $\mathbf{P}\neg\neg\alpha$ is true at w iff $\Diamond\alpha$. $\neg\mathbf{P}\neg\alpha$ is then the (weak) obligation: $\Box\alpha$.

Validity and invalidity with negation

Valid in our framework	
$\neg \mathbf{P}(\alpha \wedge \neg\alpha)$ $\mathbf{P}\neg(\alpha \wedge \beta) \leftrightarrow (\mathbf{P}\neg\alpha \vee \mathbf{P}\neg\beta)$	$\mathbf{P}\neg(\alpha \vee \beta) \leftrightarrow \mathbf{P}(\neg\alpha \wedge \neg\beta)$ $\mathbf{P}(\neg\alpha \vee \neg\beta) \rightarrow \mathbf{P}\neg(\alpha \wedge \beta)$
Invalid in our framework	
$\mathbf{P}(\alpha \vee \neg\alpha)$	$\mathbf{P}\neg(\alpha \wedge \beta) \rightarrow \mathbf{P}(\neg\alpha \vee \neg\beta)$

Mommy to baby: *You are permitted not to eat both the egg white and the egg yolk if full, but you are only permitted not to eat the egg white*, as the yolk is very good for you.

We are still working on the axiomatization of the logic with higher-order permission and action negation.

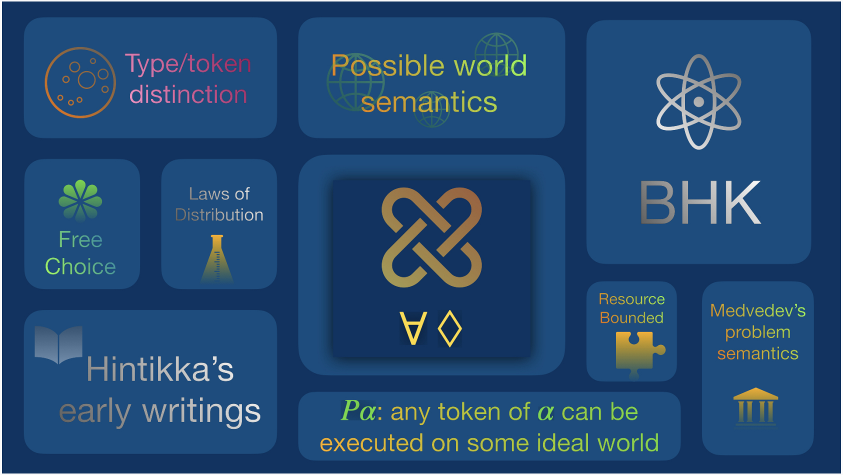
Conclusions

Conclusions

We propose a new semantic approach to deontic logic/modality based on bundled modalities. In this talk:

- We formalize strong permission as a $\forall x \Diamond$ bundle
- The propositional formulas are action types whose tokens are given by a BHK-style recursive definition
- The resulting logic admits FC and most other good properties.
- It also predicts new phenomena aligned with our linguistic intuition, which were not discussed in the literature.

Overview



Type/token distinction
Possible world semantics
BHK
Free Choice
Laws of Distribution
∀ ◊
Resource Bounded
Medvedev's problem semantics
Hintikka's early writings
$P\alpha$: any token of α can be executed on some ideal world

It is the beginning of yet another interesting story

Ongoing work

Inspired by the BHK interpretation:

- Capturing various versions of **O** and **F**
- Adding implications in the scope of modalities
- Try to solve more puzzles!

Thank you!

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