# Team-based logics around BSML

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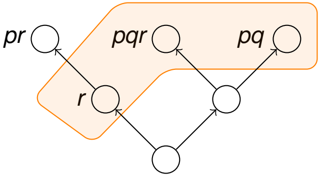
Utrecht University

The NihiL workshop
Jan. 31 - Feb. 2, 2024

# Another motivation for state-based or team-based semantics

- Bilateral state-based modal logic (BSML): Modal logic + NE
- States: sets of possible worlds.



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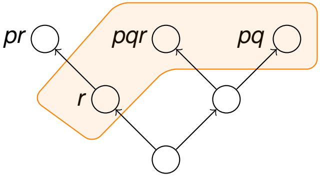

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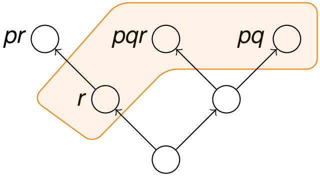


Team semantics:

- A compositional semantics introduced by Hodges (1997) for characterizing dependencies between variables,

  originally for Independence-friendly Logic (Hintikka, Sandu 1989), and later developed further in Dependence Logic (Väänänen 2007).
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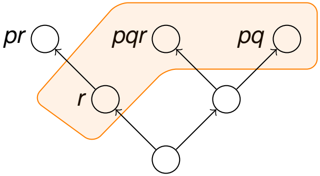


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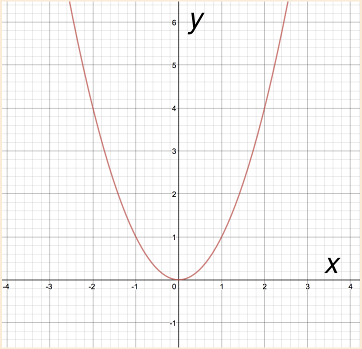
# functional dependence between variables

$$y = f(x) = x^2$$




- $x$ determines $y$

# How to characterize functional dependence between variables?

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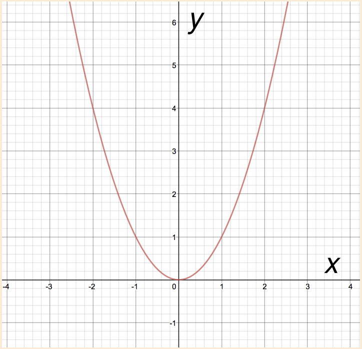


| | $x$ | $y$ | $z$ |
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| $s$ | $\sqrt{2}$ | 2 | 0 |

- $\mathbb{R} \models_s x$ `determines` $y$ iff ??

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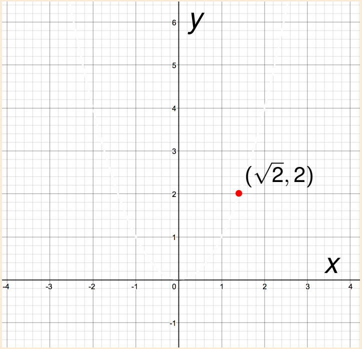


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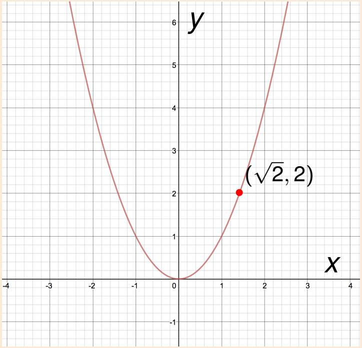


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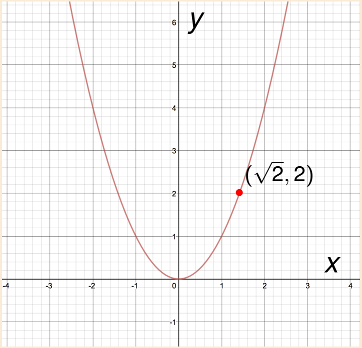


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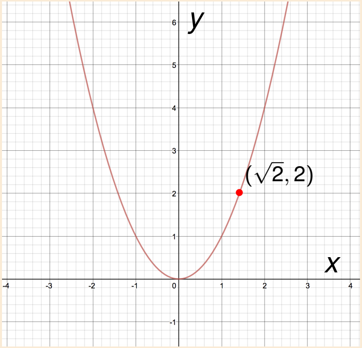


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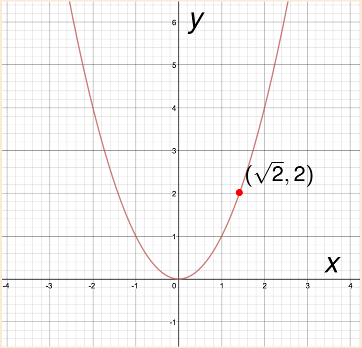


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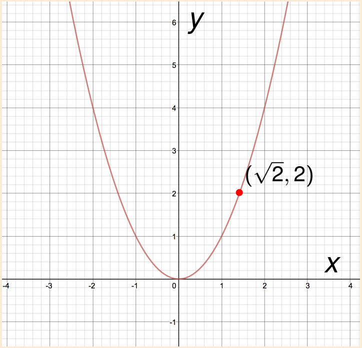


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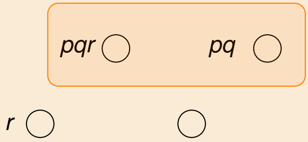
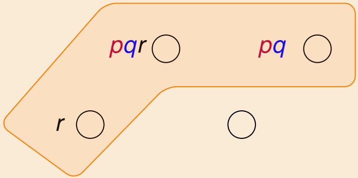



- A team $t$: a set of valuations / possible worlds
- $t \models p \texttt{ determines } q$ iff for all $u, v \in t$:

$$u(p) = v(p) \Longrightarrow u(q) = v(q)$$

– "Whether it is raining determines whether I will take my umbrella."
– "Whether this set is empty determines whether it is raining in Amsterdam."

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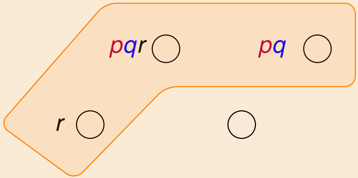
# Propositional dependence atoms

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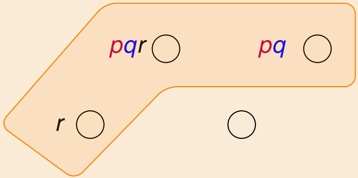


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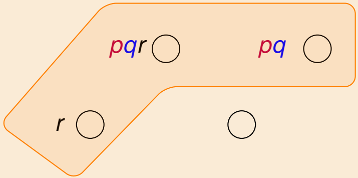


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Empty team/state property: $\emptyset \models =(\vec{p}, \vec{q})$

| | $p$ | $q$ | $r$ | $s$ |
|---|---|---|---|---|
| $v_0$ | 1 | 0 | 1 | 1 |
| $v_1$ | 1 | 0 | 1 | 0 |
| $v_2$ | 0 | 1 | 0 | 1 |
| $v_3$ | 0 | 1 | 0 | 0 |

- A team can be viewed as a relational database.
- Dependence atoms $=(\vec{p}, \vec{q})$ correspond exactly to functional dependencies $\vec{p} \to \vec{q}$ in database theory
- Armstrong's Axioms (1974) for functional dependencies:
  - $=(\vec{p}, \vec{p})$ (identity)
  - $=(\vec{p}\vec{q}, \vec{r})$ implies $=(\vec{q}\vec{p}, \vec{r})$ (commutativity)
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# Connection with database theory

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# Propositional team-based logics around BSML

Language: $\phi ::= p \mid \neg p \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid =(\vec{p}, \vec{q})$

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Recall: Propositional BSML or BSPL: $\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \mathrm{NE}$

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**Team semantics:** Let $t \subseteq 2^{\mathrm{Prop}}$ be a team/state, i.e., a set of valuations.

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- $t \models \phi \mathbin{\vee\!\!\!\vee} \psi$ iff $t \models \phi$ or $t \models \psi$
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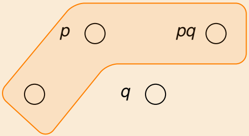


$t \not\models p$
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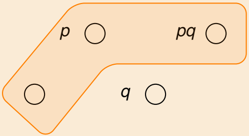


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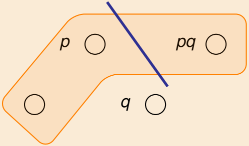


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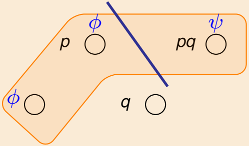


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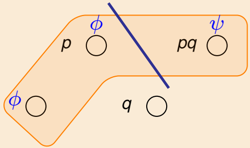


Empty Team Property: $\emptyset \models \psi$ for all $\psi$

Recall: Propositional BSML or BSPL: $\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \mathrm{NE}$

Language: $\phi ::= p \mid \neg p \mid \bot \mid \phi \wedge \phi \mid \phi \vee \phi \mid \phi \to \phi \mid \phi \mathbin{\mathbb{V}} \phi \mid {=}(\vec{p}, \vec{q})$

Local disjunction Global (or inquisitive) disjunction

Team semantics: Let $t \subseteq 2^{\text{Prop}}$ be a team/state, i.e., a set of valuations.

- $t \models p$ iff $v(p) = 1$ for all $v \in t$
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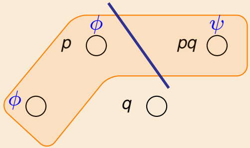


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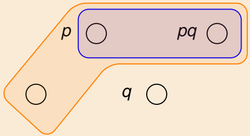


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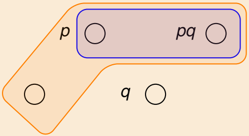


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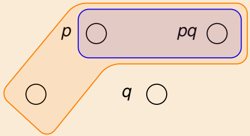


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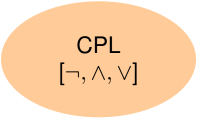
- $t \models \mathrm{NE}$ iff $t \neq \emptyset$
- $\{v\} \models \mathrm{NE}$, whereas $\emptyset \not\models \mathrm{NE}$
- Union Closure: $t \models \phi$ and $s \models \phi$ imply $t \cup s \models \phi$ for $\phi \in [\neg, \bot, \wedge, \vee, \to, \mathrm{NE}]$

# Propositional team-based logics around BSML



Downward closed

CPL
$[\neg, \wedge, \vee]$

Union closed

Downward closed

CPL
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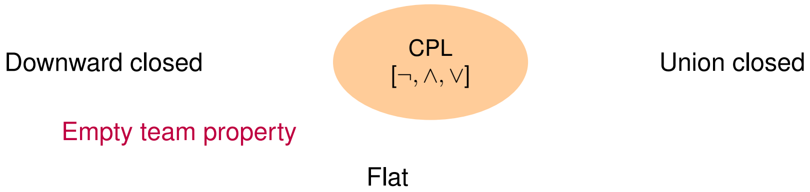
Union closed

Empty team property




- Conservativity: If $\Delta \cup \{\alpha\}$ is a set of CPL-formulas, then
$$\Delta \models_{\text{team}} \alpha \iff \Delta \models_{\text{classical}} \alpha$$
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- There are labelled sequent calculi, display calculi, and deep inference style calculi for inquisitive logic and $[\neg, \wedge, \vee, \mathbin{\vee\!\!\!\vee}]$ (Chen, Ma 2017), (Müler 2023), (Barbero, Girlando, Müler, Y. 2024), (Frittella, Greco, Palmigiano, Y. 2016), (Anttila, Iemhoff, Y. 2024)

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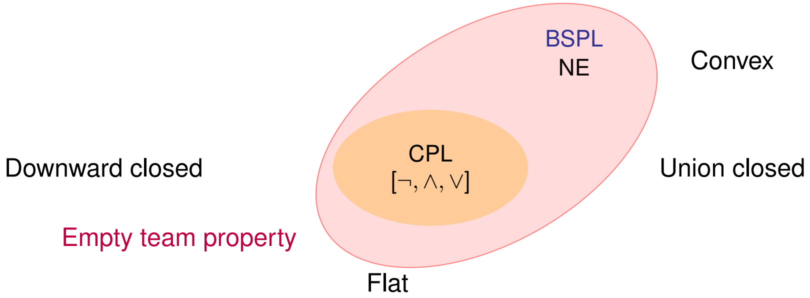


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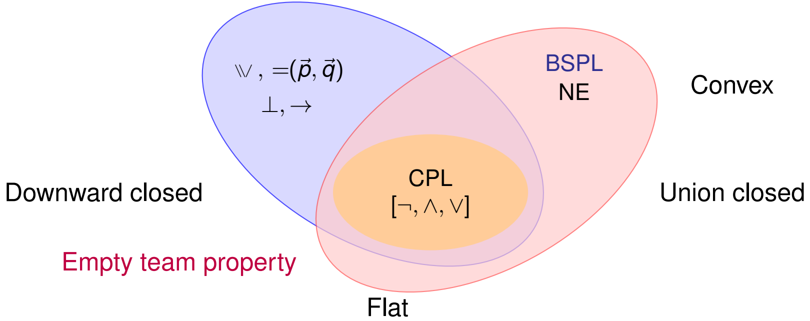


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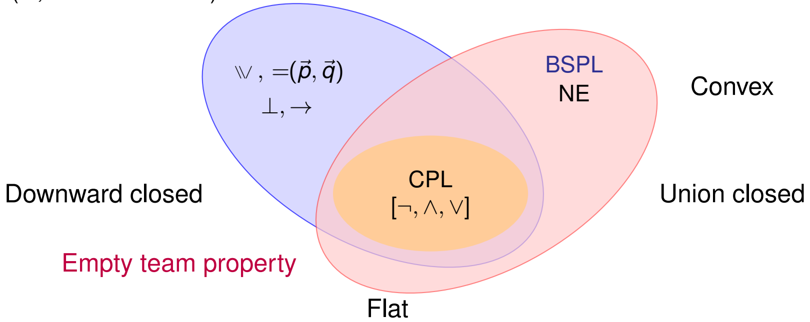


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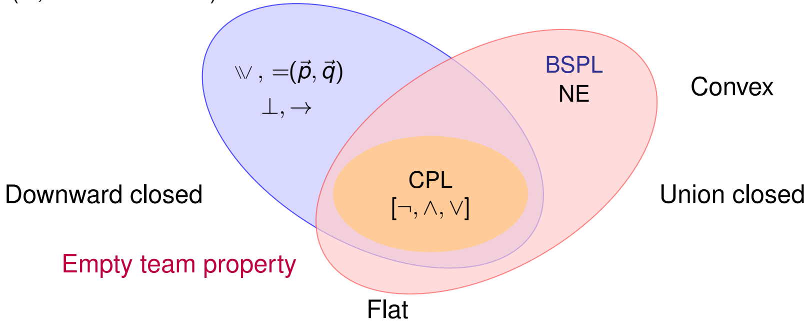


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Fix a finite set $N = \{p_1, \ldots, p_n\}$ of propositional variables.

Fact: Given a valuation $v : N \to \{0, 1\}$, the CPL-formula

$$\theta_v =$$

defines $v$, in the sense that for any $N$-valuation $u$,

$$u \models \theta_v \iff u = v$$

| $p_1$ | $p_2$ | $p_3$ |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 1 |
| 1 | 1 | 0 |

# The power of splitting teams, and the empty trap...

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$\neg p_1 \wedge p_2 \wedge \neg p_3$

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Take $\Theta_t = \bigvee_{v \in t} (p_1^{v(1)} \wedge \cdots \wedge p_n^{v(n)})$! But in fact, not exactly, due to $\emptyset$:

$$s \models \Theta_t \iff s = \bigcup_{v \in t} s_v \text{ and each } s_v \models \theta_v \iff s \subseteq t.$$

**Theorem (Y. and Väänänen 2016 )**

*The team-based logic* $[\neg, \wedge, \vee, \mathbin{\mathbb{V}}]$ *is expressively complete.*

# Expressive completeness and normal forms

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... in the following sense:

- A formula $\phi$ in $N$ defines a team property $[\![\phi]\!] = \{t \subseteq 2^N : t \models \phi\}$, which is closed downward and contains the empty team.
- For any $N$-team property P that is closed downward and contains the empty team, there is a formula $\phi \in [\neg, \wedge, \vee, \mathbin{\mathbb{V}}]$ such that $[\![\phi]\!] = \mathsf{P}$.

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  Proof. Take $\phi = \mathop{\mathbb{V}}_{t \in \mathsf{P}} \Theta_t$.

**Disjunctive normal form**

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Disjunctive normal form: "The current team $s$ is one of the teams in P"

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*Propositional dependence logic* $[\neg, \wedge, \vee, =\!(\cdot)]$ *is expressively complete.*

A conjunctive normal form: "The current team $s$ is not any team in $\overline{\mathsf{P}}$"

# Constancy atoms

- $t \models {=}(\langle\rangle, \vec{p})$ iff for all $v, u \in t$: $v(\langle\rangle) = u(\langle\rangle) \Longrightarrow v(\vec{p}) = u(\vec{p})$.

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| ... | $p$ | ... |
|---|---|---|
| ... | 1 | ... |
| ... | 1 | ... |
| ... | 1 | ... |
| ... | 1 | ... |

# Constancy atoms

- $t \models =(\vec{p})$ iff for all $v, u \in t$: $v(\vec{p}) = u(\vec{p})$.

| $\ldots$ | $p$ | $\ldots$ |
|---|---|---|
| $\ldots$ | 0 | $\ldots$ |
| $\ldots$ | 0 | $\ldots$ |
| $\ldots$ | 0 | $\ldots$ |
| $\ldots$ | 0 | $\ldots$ |

# Constancy atoms

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| $\dots$ | $p$ | $\dots$ |
|---|---|---|
| $\dots$ | 0 | $\dots$ |
| $\dots$ | 1 | $\dots$ |
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# Constancy atoms, “excluded middle"

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| $\dots$ | $p$ | $\dots$ |
|---|---|---|
| $\dots$ | 0 | $\dots$ |
| $\dots$ | 1 | $\dots$ |
| $\dots$ | 0 | $\dots$ |
| $\dots$ | 1 | $\dots$ |

Fact: $\models =\!(p) \vee =\!(p)$

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|---|---|---|
| … | 0 | … |
| … | 0 | … |
| … | 1 | … |
| … | 1 | … |

Fact: $\models =\!(p) \vee =\!(p)$

Failure of closure under uniform substitution:

$$p \vee p \models p, \text{ whereas } =\!(p) \vee =\!(p) \not\models =\!(p).$$

- $t \models \phi \vee \psi$ iff $\exists s, r \subseteq t$ s.t. $t = s \cup r$, $s \models \phi$ and $r \models \psi$

- $t \models =\!(\vec{p})$ iff for all $v, u \in t$: $v(\vec{p}) = u(\vec{p})$.

| ... | $p$ | ... |
|---|---|---|
| ... | 1 | ... |
| ... | 1 | ... |
| ... | 1 | ... |
| ... | 1 | ... |

or

| ... | $p$ | ... |
|---|---|---|
| ... | 0 | ... |
| ... | 0 | ... |
| ... | 0 | ... |
| ... | 0 | ... |

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Failure of closure under uniform substitution:

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- $t \models \phi \vee \psi$ iff $\exists s, r \subseteq t$ s.t. $t = s \cup r$, $s \models \phi$ and $r \models \psi$
- $t \models \phi \veebar \psi$ iff $t \models \phi$ or $t \models \psi$

# Constancy atoms, “excluded middle"

- $t \models =\!(\vec{p})$ iff for all $v, u \in t$: $v(\vec{p}) = u(\vec{p})$.

| ... | $p$ | ... |
|---|---|---|
| ... | 1 | ... |
| ... | 1 | ... |
| ... | 1 | ... |
| ... | 1 | ... |

or

| ... | $p$ | ... |
|---|---|---|
| ... | 0 | ... |
| ... | 0 | ... |
| ... | 0 | ... |
| ... | 0 | ... |

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Fact: $=\!(p) \equiv p \mathbin{\vee\!\!\!\vee} \neg p \equiv ?p$ (in inquisitive logic)

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Let $N = \{p_1, \ldots, p_n\}$.

**Fact:** For any $N$-team $t$,

$$t \models =\!(p_1) \wedge \cdots \wedge =\!(p_n) \iff |t| \leq 1.$$

| | $p_1$ | $p_2$ | $\ldots$ | $p_n$ |
|---|---|---|---|---|
| | 1 | 0 | $\ldots$ | 1 |
| $t$ | 0 | 1 | $\ldots$ | 0 |
| | 0 | 1 | $\ldots$ | 1 |
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# Counting with constancy atoms

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|---|---|---|---|
| 1 | 0 | $\ldots$ | 1 |
| 0 | 1 | $\ldots$ | 0 |
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| 1 | 0 | $\ldots$ | 1 |

Define

$$\eta_k = \bigvee_{i=1}^{k} (=(p_1) \wedge \cdots \wedge =(p_n))$$

Prop. For any $N$-team $t$, we have that

$$t \models \eta_k \iff |t| \leq k.$$

**Prop.** For any $N$-teams $t$, the formula $\Theta_t = \bigvee_{v \in t}(p_1^{v(1)} \wedge \cdots \wedge p_n^{v(n)})$ satisfies that for any $N$-team $s$, $\quad s \models \Theta_t \iff s \subseteq t$.

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**Proof.** Let $|t| = k + 1$. Define $\Phi_t := \eta_k \vee \Theta_{\bar{t}}$, where $\bar{t} = 2^N \setminus t$.

# Saying "no" to supersets

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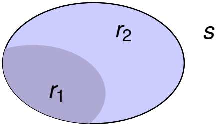

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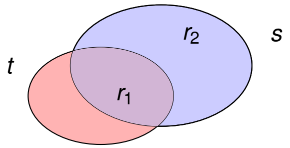

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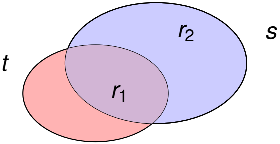


□

# Conjunctive normal form

**Theorem (Y. and Väänänen 2016 )**

*Propositional dependence logic* $[\neg, \wedge, \vee, =\!(\cdot)]$ *is expressively complete.*

Nontrivial direction: For any *N*-team property P that is closed downward and contains the empty team, there is a formula $\phi \in [\neg, \wedge, \vee, =\!(\cdot)]$ such that $[\![\phi]\!] = \mathsf{P}$.

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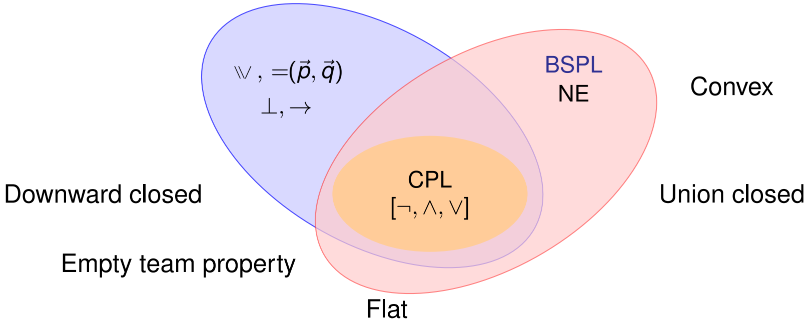
Conjunctive normal form: "The current team $s$ is not any team in $\overline{\mathsf{P}}$"

Disjunctive normal form: "The current team $s$ is one of the teams in P"

$\mathbb{W}_{t \in \mathsf{P}} \Theta_t$

# Union closed team-based logics around BSML?



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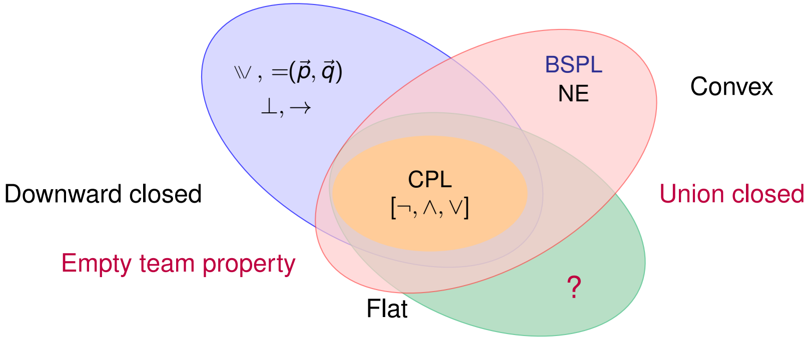

# Database theory connection revisited: inclusion dependencies

- A team can be viewed as a relational database.
- Dependence atoms $=\!(\vec{p}, \vec{q})$ correspond to functional dependencies $\vec{p} \to \vec{q}$ in database theory
- Inclusion dependencies give rise to inclusion atoms (Galliani 2012):

| $p$ | $q$ | $r$ | $s$ |
|---|---|---|---|
| 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 1 |
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|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 1 | 1 |
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| $\bot$ | $\top$ | $p$ | $q$ | $r$ | $s$ |
|---|---|---|---|---|---|
| 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 | 0 |

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We now consider the logic $[\neg, \wedge, \vee, \vec{a} \subseteq \vec{b}]$, known as propositional inclusion logic.

Empty team property: $\emptyset \models \psi$ for all $\psi$

Union closure: If $t \models \phi$ and $s \models \phi$, then $t \cup s \models \phi$.

**Theorem ((Y. 2022), (Hella, Kuusisto, Meier, Vollmer 2019))**

*Propositional inclusion logic* $[\neg, \wedge, \vee, \subseteq]$ *is expressively complete over union closed team properties that contain the empty team.*

Proof idea of nontrivial direction: For any $N$-team property P that is union closed and contains the empty team, construct a formula $\phi \in [\neg, \wedge, \vee, \subseteq]$ such that $[\![\phi]\!] = \mathsf{P}$, i.e., $s \models \phi \iff s \in \mathsf{P}$.

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Thus, $s \models \Theta_t \wedge \Psi_t \iff s = t \text{ or } s = \emptyset$.

# Defining supersets (... but ignore the empty set please)

Prop. For any $N$-team $t$, there is a formula $\Psi_t$ s.t. for any $N$-team $s$,

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| $p_1$ | $p_2$ | $p_3$ |
|---|---|---|
| 1 | 0 | 1 |
| 0 | 1 | 0 |
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$$s \models \underline{v(p_1)} \ldots \underline{v(p_n)} \subseteq p_1 \ldots p_n \iff v \in s \text{ or } s = \varnothing,$$

where $\underline{1} := \top$ and $\underline{0} := \bot$.

| | $p_1$ | $p_2$ | $p_3$ | |
|---|---|---|---|---|
| | 1 | 0 | 1 | |
| $v$ | 0 | 1 | 0 | $\bot\top\bot \subseteq p_1 p_2 p_3$ |
| | 1 | 1 | 1 | |
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For any $N$-team $t$, define

$$\Psi_t = \bigwedge_{v \in t} \underline{v(p_1)} \dots \underline{v(p_n)} \subseteq p_1 \dots p_n.$$

...

□

| | $p_1$ | $p_2$ | $p_3$ | |
|---|---|---|---|---|
| | 1 | 0 | 1 | |
| $v$ | 0 | 1 | 0 | $\bot\top\bot \subseteq p_1 p_2 p_3$ |
| | 1 | 1 | 1 | |
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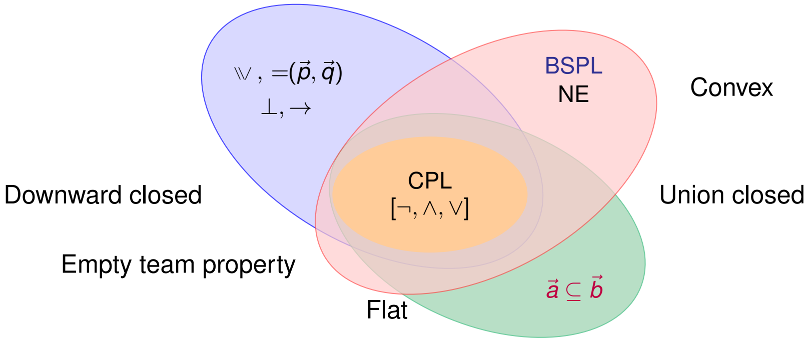
# More expressively complete logics?



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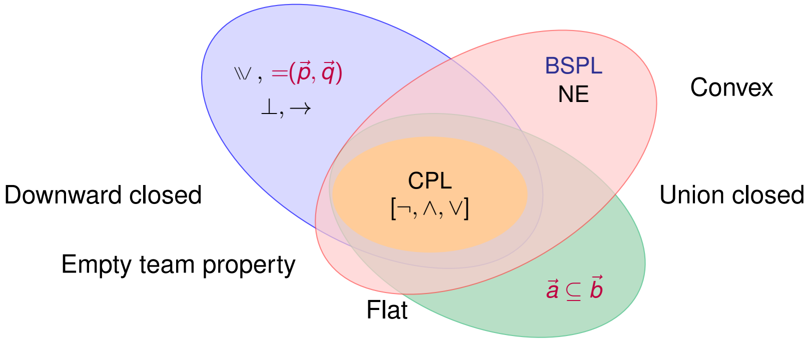


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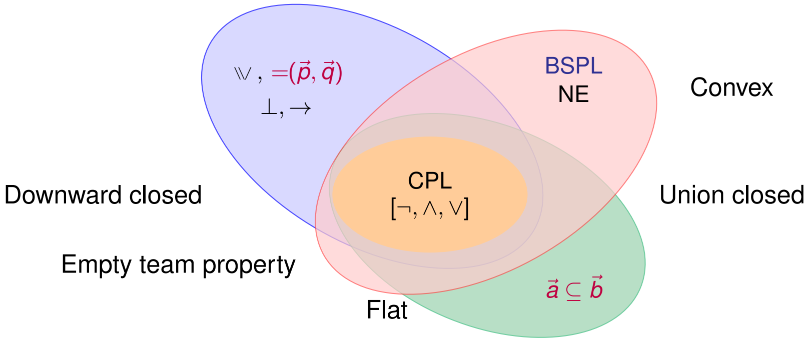


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Background:

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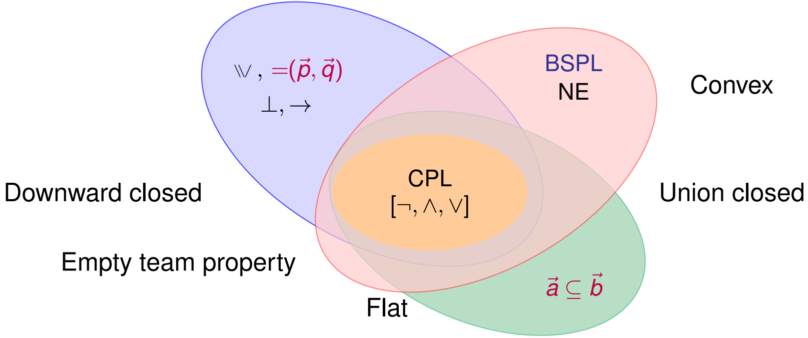


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# Expressive completeness and conjunctive normal form

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To be precise:

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| | $p_1$ | $p_2$ | $p_3$ | |
|---|---|---|---|---|
| | 1 | 0 | 1 | |
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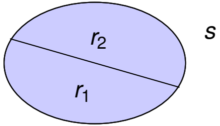

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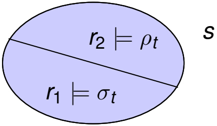

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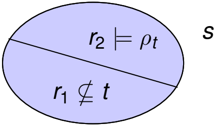

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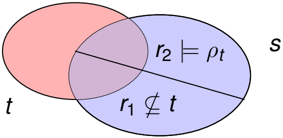

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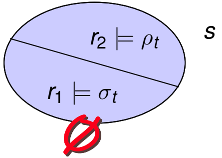

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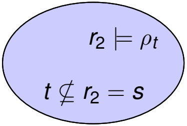


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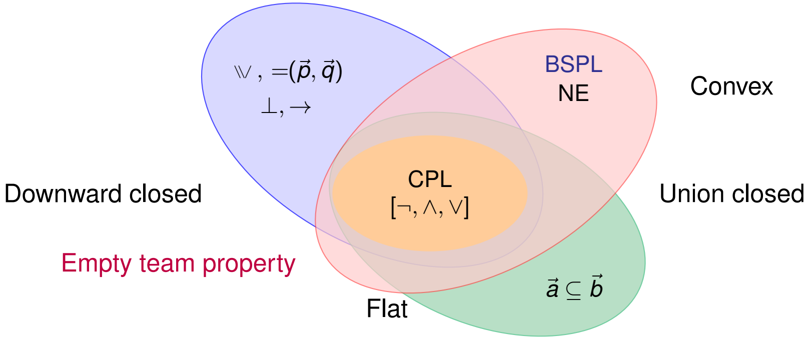


Summary:

- We have discussed a number of expressively complete propositional team-based logics around BSPL (that have the empty team property).
- These results can also be generalized to the modal logic setting.
- (More) applications?