Beyond sequent calculus: proof systems for conditional logics

Marianna Girlando

ILLC, University of Amsterdam

Based on joint work with: Björn Lellmann, Sara Negri, Nicola Olivetti and Gian Luca Pozzato

The Nihil Workshop, Amsterdam
02 February 2024

Why (analytic) proof systems?

Why (analytic) proof systems?

- Clear understanding of the logic

Why (analytic) proof systems?

- Clear understanding of the logic
- Establish properties of the logic

Why (analytic) proof systems?

- Clear understanding of the logic
- Establish properties of the logic

  Decidability and FMP

Why (analytic) proof systems?

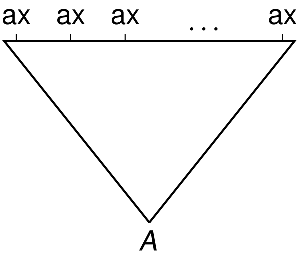
- Clear understanding of the logic
- Establish properties of the logic

  Decidability and FMP   Is A valid?

Why (analytic) proof systems?

- Clear understanding of the logic
- Establish properties of the logic

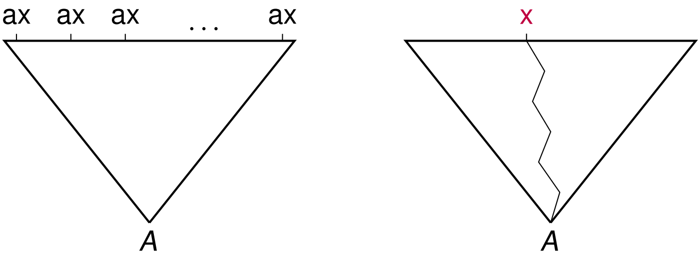
Decidability and FMP   Is *A* valid?



Why (analytic) proof systems?

- Clear understanding of the logic
- Establish properties of the logic

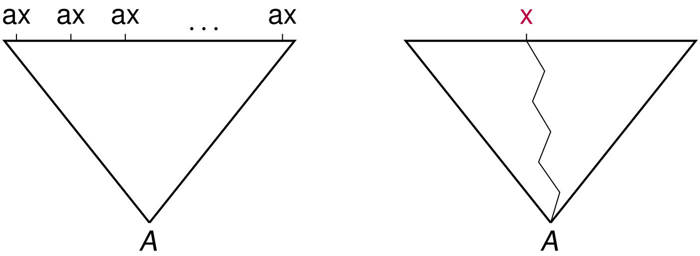
Decidability and FMP   Is *A* valid?



Why (analytic) proof systems?

- Clear understanding of the logic
- Establish properties of the logic

Decidability and FMP   Is *A* valid?

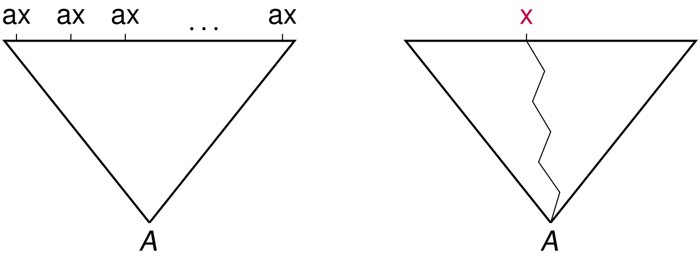


Intuitionistic S4: [G, Kuznets, Marin, Morales, Straßburger, 2023]

Why (analytic) proof systems?

- Clear understanding of the logic
- Establish properties of the logic

Decidability and FMP   Is A valid?



Intuitionistic S4: [G, Kuznets, Marin, Morales, Straßburger, 2023]

Interpolation
[Kuznets, Lellmann, 2018], [van der Giessen, Jalali, Kuznets, 2023]

Outline

- Conditional logics
- Semantics
- Proof theory for conditional logics
  - Labelled calculi for conditional logics
  - Sequent calculi with blocks for (some) Lewis' logics

Conditional logics

Conditionals in natural language

If A *then* B

If A then B

- *If I hadn't overslept, then I would have caught the train.*

Conditionals in natural language

If *A* *then* *B*

- *If I hadn't overslept, then I would have caught the train.*
- *If Tux is a bird then it can fly.*

Conditionals in natural language

If A then B

- *If I hadn't overslept, then I would have caught the train.*
- *If Tux is a bird then it can fly.*
  But if Tux is a bird and a penguin, then it can't fly.

Conditionals in natural language

If A then B

- *If I hadn't overslept, then I would have caught the train.*
- *If Tux is a bird then it can fly.*
  But if Tux is a bird and a penguin, then it can't fly.
   *Normally, birds can fly*.

Paradoxes of material implication

Paradoxes of material implication

- *If I hadn't overslept, then I would have caught the train.*

- *If I hadn't overslept, then I would have caught the train.*

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

Paradoxes of material implication

- *If I hadn't overslept, then I would have caught the train.*

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

- *If Tux is a bird then it can fly.*
  But if Tux is a bird and a penguin, then it can't fly.

Paradoxes of material implication

- *If I hadn't overslept, then I would have caught the train.*

A	B	$A \to B$
T	T	T
T	F	F
F	T	T
F	F	T

- *If Tux is a bird then it can fly.*
  But if Tux is a bird and a penguin, then it can't fly.

$$\textit{Monotonicity} \quad (A \to B) \to ((A \wedge C) \to B)$$

Conditionals in a modal framework

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B$$

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \to B$$
$$\neg A := A \to \bot$$

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \to B \mid \Box A$$
$$\neg A := A \to \bot$$

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid \Box A$$
$$\neg A := A \rightarrow \bot$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess . . .

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \to B \mid \Box A$$
$$\neg A := A \to \bot$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess ...

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \to B$$

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid \Box A$$
$$\neg A := A \rightarrow \bot$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess ...

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid A > B$$

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid \Box A$$
$$\neg A := A \rightarrow \bot$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess ...

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid A > B$$

$$\Box A := \neg A > \bot$$

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid \Box A$$

$$\neg A := A \rightarrow \bot$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess . . .

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid A > B$$

$$\Box A := \neg A > \bot$$

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid A \preccurlyeq B$$

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid \Box A$$

$$\neg A := A \rightarrow \bot$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess ...

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid A > B$$

$$\Box A := \neg A > \bot$$

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid A \preccurlyeq B$$

"A is at least as plausible as B"

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \to B \mid \Box A$$

$$\neg A := A \to \bot$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess . . .

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \to B \mid A > B$$

$$\Box A := \neg A > \bot$$

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \to B \mid A \preccurlyeq B$$

"A is at least as plausible as B"

$$\Box A := \bot \preccurlyeq \neg A$$

$$A > B := (\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \vee B))$$

Conditionals in a modal framework

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid \Box A$$

$$\neg A := A \rightarrow \bot$$

1960-70: Stalnaker, Lewis, Nute, Chellas, Burgess ...

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid A > B$$

$$\Box A := \neg A > \bot$$

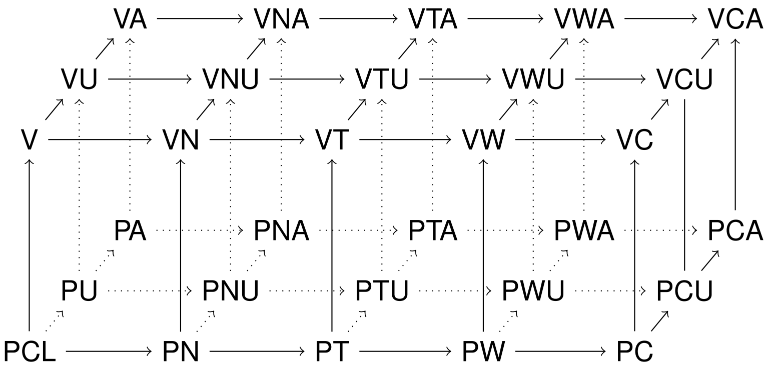
$$A \preccurlyeq B := ((A \vee B) > \bot) \vee ((A \vee B) > \neg A)$$

$$A, B ::= p \mid \bot \mid A \vee B \mid A \wedge B \mid A \rightarrow B \mid A \preccurlyeq B$$

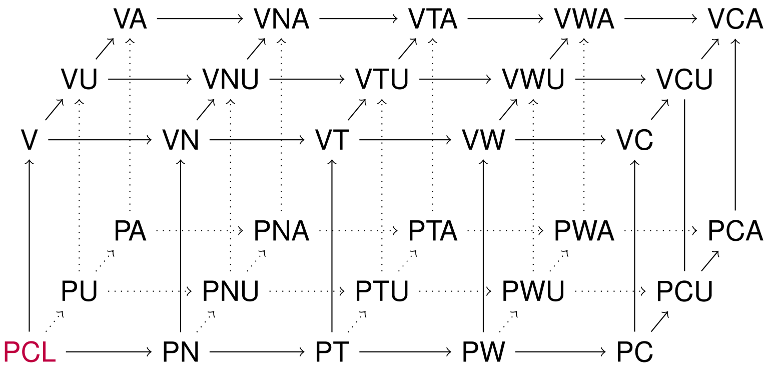
"A is at least as plausible as B"

$$\Box A := \bot \preccurlyeq \neg A$$

$$A > B := (\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \vee B))$$

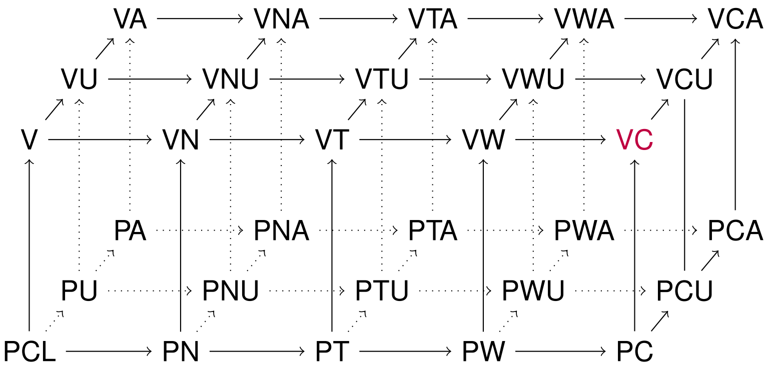

VA
VNA
VTA
VWA
VCA
VU
VNU
VTU
VWU
VCU
V
VN
VT
VW
VC
PA
PNA
PTA
PWA
PCA
PU
PNU
PTU
PWU
PCU
PCL
PN
PT
PW
PC

Conditional logics



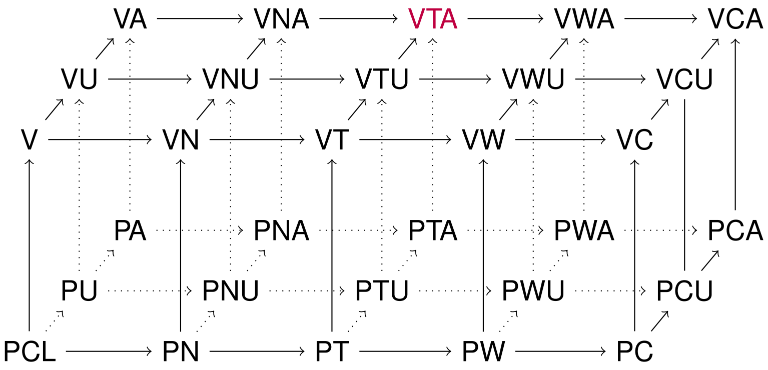
- Prototypical properties [KLM, 1990]

Conditional logics



- Prototypical properties [KLM, 1990]
- Counterfactuals [Lewis,1973]

Conditional logics



- Prototypical properties [KLM, 1990]
- Counterfactuals [Lewis,1973]
- Conditional belief of agents [Baltag and Smets, 2006, 2008]

Axioms

PCL: classical propositional logic plus

rcea $\dfrac{(A > C) \leftrightarrow (B > C)}{A \leftrightarrow B}$ rck $\dfrac{(C > A) \rightarrow (C > B)}{A \rightarrow B}$

id $A > A$

r.and $(A > B) \wedge (A > C) \rightarrow (A > (B \wedge C))$

cm $(A > C) \wedge (A > B) \rightarrow ((A \wedge B) > C)$

rt $(A > B) \wedge ((A \wedge B) > C) \rightarrow (A > C)$

or $(A > C) \wedge (B > C) \rightarrow ((A \vee B) > C)$

Axioms

PCL: classical propositional logic plus

rcea $\dfrac{(A > C) \leftrightarrow (B > C)}{A \leftrightarrow B}$ rck $\dfrac{(C > A) \rightarrow (C > B)}{A \rightarrow B}$

id $A > A$

r.and $(A > B) \wedge (A > C) \rightarrow (A > (B \wedge C))$

cm $(A > C) \wedge (A > B) \rightarrow ((A \wedge B) > C)$

rt $(A > B) \wedge ((A \wedge B) > C) \rightarrow (A > C)$

or $(A > C) \wedge (B > C) \rightarrow ((A \vee B) > C)$

V: PCL plus

cv $(A > C) \wedge \neg(A > \neg B) \rightarrow ((A \wedge B) > C)$

Axioms

PCL: classical propositional logic plus

rcea $\dfrac{(A > C) \leftrightarrow (B > C)}{A \leftrightarrow B}$ rck $\dfrac{(C > A) \rightarrow (C > B)}{A \rightarrow B}$

id $A > A$

r.and $(A > B) \wedge (A > C) \rightarrow (A > (B \wedge C))$

cm $(A > C) \wedge (A > B) \rightarrow ((A \wedge B) > C)$

rt $(A > B) \wedge ((A \wedge B) > C) \rightarrow (A > C)$

or $(A > C) \wedge (B > C) \rightarrow ((A \vee B) > C)$

V: PCL plus

cv $(A > C) \wedge \neg(A > \neg B) \rightarrow ((A \wedge B) > C)$

Extensions of PCL and V

n $\neg(\top > \bot)$

w $(A > B) \rightarrow (A \rightarrow B)$

u_1 $(\neg A > \bot) \rightarrow (\neg(\neg A > \bot) > \bot)$

a_1 $(A > B) \rightarrow (C > (A > B))$

t $A \rightarrow \neg(A > \bot)$

c $(A \wedge B) \rightarrow (A > B)$

u_2 $\neg(A > \bot) \rightarrow ((A > \bot) > \bot)$

a_2 $\neg(A > B) \rightarrow (C > \neg(A > B))$

Semantics

Several kinds of possible-world semantics

Several kinds of possible-world semantics

- Sphere semantics [Lewis,1973]

Several kinds of possible-world semantics

- Sphere semantics [Lewis,1973]
- Preferential semantics [Burgess,1981]

Several kinds of possible-world semantics

- Sphere semantics [Lewis,1973]
- Preferential semantics [Burgess,1981]
- Selection function semantics [Chellas, 1975]

Several kinds of possible-world semantics

- Sphere semantics [Lewis,1973]
- Preferential semantics [Burgess,1981]
- Selection function semantics [Chellas, 1975]
- Neighbourhood semantics [Scott, 1970, Montague, 1970]

Several kinds of possible-world semantics

- Sphere semantics [Lewis,1973]
- Preferential semantics [Burgess,1981]
- Selection function semantics [Chellas, 1975]
- Neighbourhood semantics [Scott, 1970, Montague, 1970]

  Direct proof of soundness and completeness w.r.t. the axiomatization of PCL and extensions
  [G, Negri, Olivetti, 2021]

Neighbourhood models for VC

$$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$$

Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$

y

x

z

k

Neighbourhood models for VC

$$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W)) \text{ s.t. } \emptyset \notin N(x)$$

y

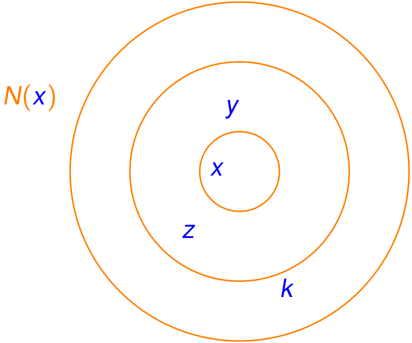
x

z

k

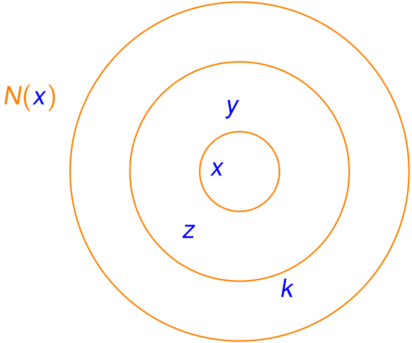
Neighbourhood models for VC

$$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W)) \text{ s.t. } \emptyset \notin N(x)$$



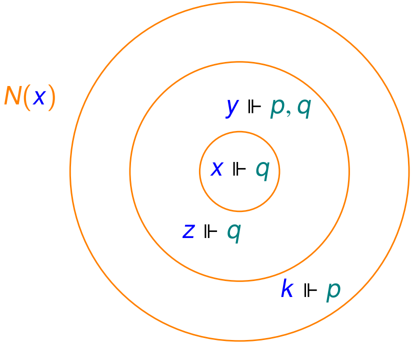
Neighbourhood models for VC

$$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \rightarrow \mathcal{P}(\mathcal{P}(W)) \text{ s.t. } \emptyset \notin N(x) \quad [\![\cdot]\!] : Atm \rightarrow \mathcal{P}(W)$$



Neighbourhood models for VC

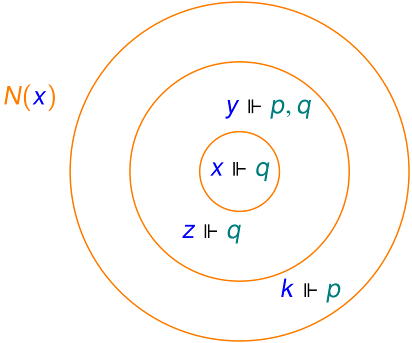
$$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W)) \text{ s.t. } \emptyset \notin N(x) \quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$$



Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$ $\quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x)$ $\quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting for all x , for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

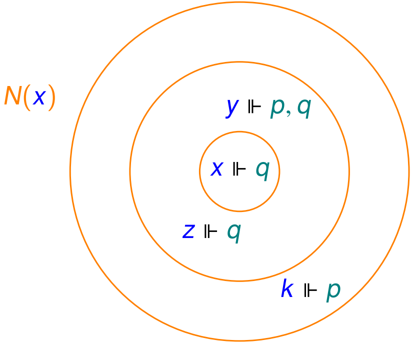


Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$ $N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x)$ $[\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting   for all x , for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

Centering   for all x , for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$

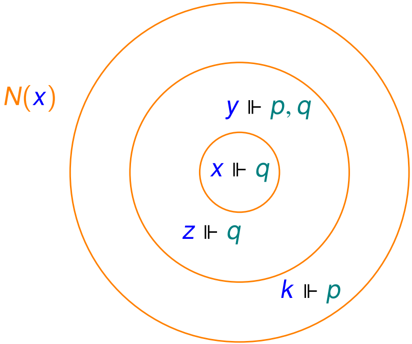


Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x) \quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting for all x , for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

Centering for all x , for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$



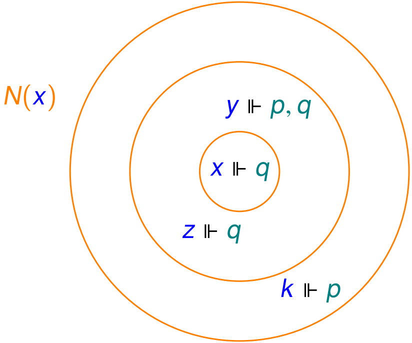
$x \Vdash q \preccurlyeq p$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} p$ then $\alpha \Vdash^{\exists} q$

Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle$ $\quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x)$ $\quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting for all x, for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

Centering for all x, for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$



$x \Vdash q \preccurlyeq p$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} p$ then $\alpha \Vdash^{\exists} q$

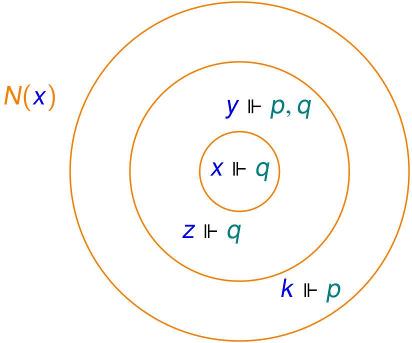
$\alpha \Vdash^{\forall} A \equiv \forall y \in \alpha, y \Vdash A$ $\qquad$ $\alpha \Vdash^{\exists} A \equiv \exists y \in \alpha$ s. t. $y \Vdash A$

Neighbourhood models for VC

$\mathcal{M} = \langle W, N, [\![\cdot]\!] \rangle \quad N : W \to \mathcal{P}(\mathcal{P}(W))$ s.t. $\emptyset \notin N(x) \quad [\![\cdot]\!] : Atm \to \mathcal{P}(W)$

Nesting for all x, for all $\alpha, \beta \in N(x)$, $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$

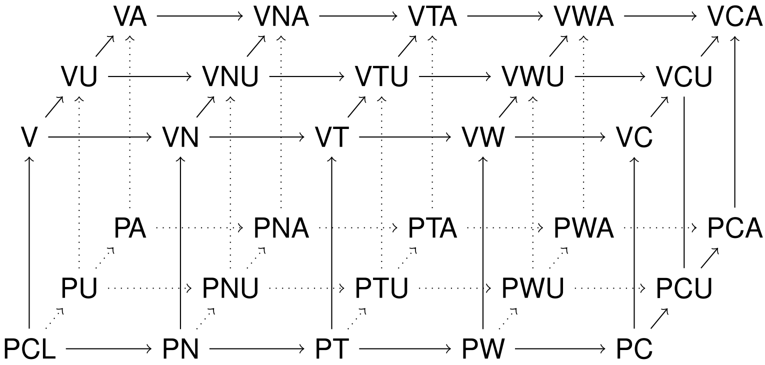
Centering for all x, for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$



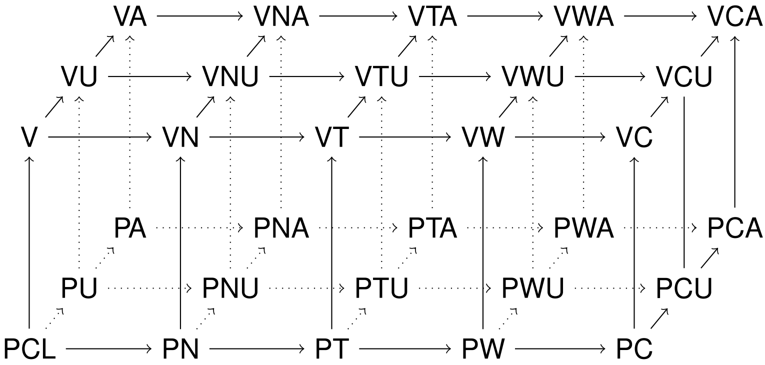
$x \Vdash p > q$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} p$, then there is $\beta \in N(x)$ s.t. $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} p$ and $\beta \Vdash^{\forall} p \to q$

$\alpha \Vdash^{\forall} A \equiv \forall y \in \alpha, y \Vdash A$ $\qquad$ $\alpha \Vdash^{\exists} A \equiv \exists y \in \alpha$ s. t. $y \Vdash A$

Conditional logics

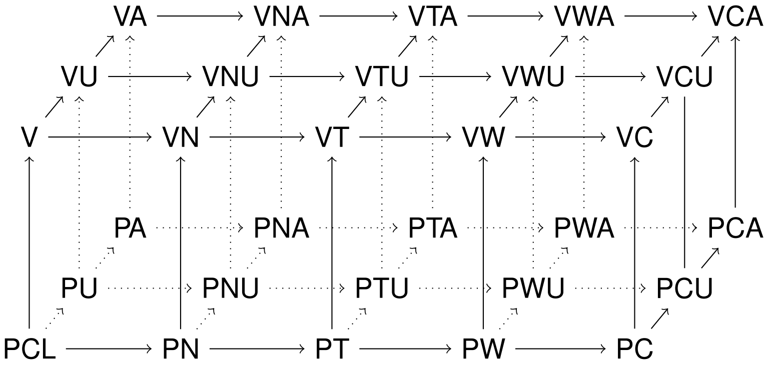


Conditional logics



★ *Nesting* For all x, for all $\alpha, \beta \in N(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. V

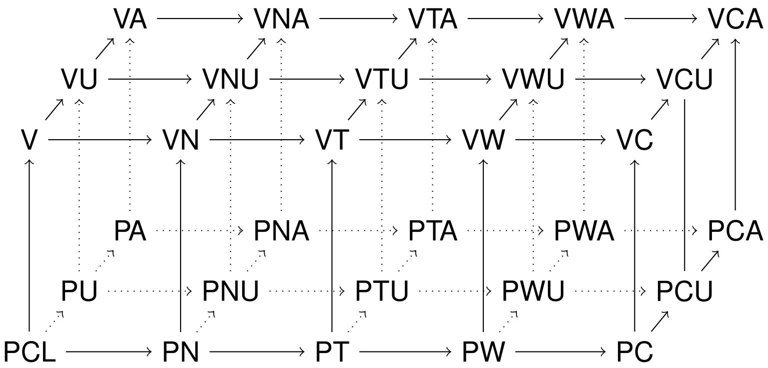
Conditional logics



★ *Centering* For all x, for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$. C

★ *Nesting* For all x, for all $\alpha, \beta \in N(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. V

Conditional logics



- ⋆ *Normality* For all x, $N(x) \neq \emptyset$. N
- ⋆ *Total reflexivity* For all x, there is $\alpha \in N(x)$ such that $x \in \alpha$. T
- ⋆ *Weak centering* For all x, $N(x) \neq \emptyset$ and for all $\alpha \in N(x)$, $x \in \alpha$. W
- ⋆ *Centering* For all x, for all $\alpha \in N(x)$, $x \in \alpha$ and $\{x\} \in N(x)$. C
- ⋆ *Uniformity* For all x, y, $\bigcup N(y) = \bigcup N(x)$. U
- ⋆ *Absoluteness* For all x, y, $N(x) = N(y)$. A
- ⋆ *Nesting* For all x, for all $\alpha, \beta \in N(x)$, either $\alpha \subseteq \beta$ or $\beta \subseteq \alpha$. V

Proof systems for conditional logics

Sequent calculus for modal logic

Sequent calculus for propositional logic [Gentzen, 1933-34]

Sequent calculus for propositional logic [Gentzen, 1933-34]

Γ, Δ multisets of formulas $\qquad \Gamma \Rightarrow \Delta \rightsquigarrow \bigwedge \Gamma \rightarrow \bigvee \Delta$

Sequent calculus for propositional logic [Gentzen, 1933-34]

Γ, Δ multisets of formulas $\quad \Gamma \Rightarrow \Delta \rightsquigarrow \bigwedge \Gamma \to \bigvee \Delta$

$$\text{init}\,\frac{}{p, \Gamma \Rightarrow \Delta, p} \quad \bot\,\frac{}{\bot, \Gamma \Rightarrow \Delta} \quad \to_L\,\frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \quad \to_R\,\frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

Sequent calculus for modal logic

Sequent calculus for propositional logic [Gentzen, 1933-34]

Γ, Δ multisets of formulas $\qquad \Gamma \Rightarrow \Delta \rightsquigarrow \bigwedge \Gamma \to \bigvee \Delta$

$$\text{init}\ \frac{}{p, \Gamma \Rightarrow \Delta, p} \quad \bot\ \frac{}{\bot, \Gamma \Rightarrow \Delta} \quad \to_{\mathsf{L}}\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \quad \to_{\mathsf{R}}\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

Proof systems for modal logics

Sequent calculus for propositional logic [Gentzen, 1933-34]

Γ, Δ multisets of formulas $\quad \Gamma \Rightarrow \Delta \rightsquigarrow \bigwedge \Gamma \to \bigvee \Delta$

$$\text{init}\ \frac{}{p, \Gamma \Rightarrow \Delta, p} \qquad \bot\ \frac{}{\bot, \Gamma \Rightarrow \Delta} \qquad \to_L\ \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \to B, \Gamma \Rightarrow \Delta} \qquad \to_R\ \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B}$$

Proof systems for modal logics $\rightsquigarrow$ Adding modal rules:

$$\Box\ \frac{\Sigma \Rightarrow A}{\Box\Sigma, \Gamma \Rightarrow \Delta, \Box A}$$

$\Box\Sigma = \Box B_1, \ldots, \Box B_k$, for $0 \leq k$

Sequent calculus for modal logic

Sequent calculus for propositional logic [Gentzen, 1933-34]

Γ, Δ multisets of formulas $\qquad \Gamma \Rightarrow \Delta \rightsquigarrow \bigwedge \Gamma \rightarrow \bigvee \Delta$

$$\text{init}\ \frac{}{p, \Gamma \Rightarrow \Delta, p} \qquad \bot\ \frac{}{\bot, \Gamma \Rightarrow \Delta} \qquad \rightarrow_L \frac{\Gamma \Rightarrow \Delta, A \quad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \rightarrow_R \frac{A, \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B}$$

Proof systems for modal logics $\rightsquigarrow$ Adding modal rules:

$$\Box\ \frac{\Sigma \Rightarrow A}{\Box\Sigma, \Gamma \Rightarrow \Delta, \Box A}$$

$\Box\Sigma = \Box B_1, \ldots, \Box B_k$, for $0 \leq k$

Problem for some systems of modal logics (S5), no **cut-free** Gentzen-style sequent calculus is known

$$\text{cut}\ \frac{\Gamma \Rightarrow \Delta, A \quad A, \Gamma' \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Solutions

Solutions

☞ Enrich the language of the calculus

☞ Enrich the language of the calculus

Labelled calculus [Negri, 2005]

Solutions

☞ Enrich the language of the calculus

Labelled calculus [Negri, 2005]

$xRy \rightsquigarrow$ "x has access to y"

Solutions

☞ Enrich the language of the calculus

Labelled calculus [Negri, 2005]

$xRy \rightsquigarrow$ "x has access to y"

$x : A \rightsquigarrow$ "x satisfies A"

Solutions

- ☞ Enrich the language of the calculus

  Labelled calculus [Negri, 2005]

  xRy ⇝ "x has access to y"

  $x : A$ ⇝ "x satisfies A"

- ☞ Enrich the structure of sequents

Solutions

☞ Enrich the language of the calculus

Labelled calculus [Negri, 2005]

$xRy \rightsquigarrow$ "x has access to y"

$x : A \rightsquigarrow$ "x satisfies A"

☞ Enrich the structure of sequents

- Hypersequent calculus [Avron, 1996]

Solutions

☞ Enrich the language of the calculus

Labelled calculus [Negri, 2005]

xRy ⇝ "x has access to y"

$x : A$ ⇝ "x satisfies A"

☞ Enrich the structure of sequents

- Hypersequent calculus [Avron, 1996]

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

Solutions

☞ Enrich the language of the calculus

Labelled calculus [Negri, 2005]

xRy ⇝ "x has access to y"

$x : A$ ⇝ "x satisfies A"

☞ Enrich the structure of sequents

- Hypersequent calculus [Avron, 1996]

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

- Nested sequent calculi [Brünnler, 2009]

Solutions

☞ Enrich the language of the calculus

Labelled calculus [Negri, 2005]

$xRy \rightsquigarrow$ "x has access to y"

$x : A \rightsquigarrow$ "x satisfies A"

☞ Enrich the structure of sequents

- Hypersequent calculus [Avron, 1996]

$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

- Nested sequent calculi [Brünnler, 2009]

$$\Delta, [\Delta_1], \ldots, [\Delta_n]$$

Solutions

☞ Enrich the language of the calculus

Labelled calculus [Negri, 2005]

$xRy \rightsquigarrow$ "x has access to y"

$x : A \rightsquigarrow$ "x satisfies A"

☞ Enrich the structure of sequents

- Hypersequent calculus [Avron, 1996]

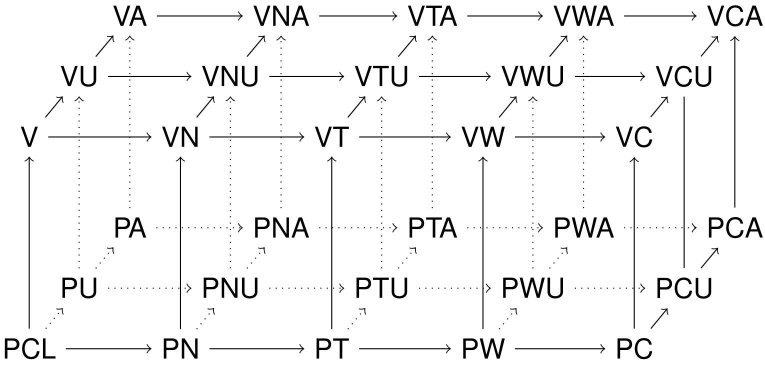
$$\Gamma_1 \Rightarrow \Delta_1 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

- Nested sequent calculi [Brünnler, 2009]

$$\Delta, [\Delta_1], \ldots, [\Delta_n]$$

- ...

Labelled calculi for conditional logics



Enriching the language: modal logics

Enriching the language: modal logics

- ☞ Countably many variables for worlds: $x, y, z \ldots$ (labels)

Enriching the language: modal logics

- ☞ Countably many variables for worlds: $x, y, z \ldots$ (labels)
- ☞ Labelled formulas

Enriching the language: modal logics

- ☞ Countably many variables for worlds: $x, y, z \ldots$ (labels)
- ☞ Labelled formulas
  - ▹ $xRy \rightsquigarrow$ "x has access to y" (relational atoms)

Enriching the language: modal logics

- ☞ Countably many variables for worlds: $x, y, z \ldots$ (labels)
- ☞ Labelled formulas
  - ▹ $xRy \rightsquigarrow$ "x has access to y" (relational atoms)
  - ▹ $x : A \rightsquigarrow$ "x satisfies A" (labelled formulas)

Enriching the language: modal logics

- ☞ Countably many variables for worlds: $x, y, z \ldots$ (labels)
- ☞ Labelled formulas
  - ▹ $xRy \rightsquigarrow$ "x has access to y" (relational atoms)
  - ▹ $x : A \rightsquigarrow$ "x satisfies A" (labelled formulas)
- ☞ Labelled sequent: $\mathcal{R}, \Gamma \Rightarrow \Delta$

Enriching the language: modal logics

- ☞ Countably many variables for worlds: $x, y, z \ldots$ (labels)
- ☞ Labelled formulas
  - ▹ $xRy \rightsquigarrow$ "x has access to y" (relational atoms)
  - ▹ $x : A \rightsquigarrow$ "x satisfies A" (labelled formulas)
- ☞ Labelled sequent: $\mathcal{R}, \Gamma \Rightarrow \Delta$
- ☞ Rules for $\Box$

$$\Box_L \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A} y! \qquad \Box_R \frac{xRy, \mathcal{R}, x : \Box A, y : A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta}$$

$$x \Vdash \Box A \quad \textit{iff} \quad \textit{for all } y \textit{ s.t. } xRy, y \Vdash A$$

Enriching the language: modal logics

- ☞ Countably many variables for worlds: $x, y, z \dots$ (labels)
- ☞ Labelled formulas
  - ▹ $xRy \rightsquigarrow$ "x has access to y" (relational atoms)
  - ▹ $x : A \rightsquigarrow$ "x satisfies A" (labelled formulas)
- ☞ Labelled sequent: $\mathcal{R}, \Gamma \Rightarrow \Delta$
- ☞ Rules for $\Box$

$$\Box_L \frac{xRy, \mathcal{R}, \Gamma \Rightarrow \Delta, y : A}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : \Box A} y! \qquad \Box_R \frac{xRy, \mathcal{R}, x : \Box A, y : A, \Gamma \Rightarrow \Delta}{xRy, \mathcal{R}, x : \Box A, \Gamma \Rightarrow \Delta}$$

$$x \Vdash \Box A \quad \textit{iff} \quad \textit{for all } y \textit{ s.t. } xRy, y \Vdash A$$

- ☞ Rules for frame conditions, example: transitivity

$$\text{tr} \frac{xRz, xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}{xRy, yRz, \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Enriching the language: conditional logics

☞ Countably many variables for worlds $x, y, z, \ldots$

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \ldots$

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \ldots$

- ☞ Relational atoms

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \dots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \dots$

- ☞ Relational atoms
  - ▹ $x \in a \rightsquigarrow$ "x is an element of a"

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \dots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \dots$

- ☞ Relational atoms
  - ▹ $x \in a$ $\rightsquigarrow$ "x is an element of a"
  - ▹ $a \in N(x)$ $\rightsquigarrow$ "a is an element of $N(x)$"

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \ldots$

- ☞ Relational atoms
  - $x \in a \rightsquigarrow$ "x is an element of a"
  - $a \in N(x) \rightsquigarrow$ "a is an element of $N(x)$"
  - $a \subseteq b \rightsquigarrow$ "a is included in b"

Enriching the language: conditional logics

- Countably many variables for worlds $x, y, z, \ldots$
- Countably many variables for neighbourhoods $a, b, c, \ldots$

- Relational atoms
  - $x \in a \rightsquigarrow$ "x is an element of a"
  - $a \in N(x) \rightsquigarrow$ "a is an element of $N(x)$"
  - $a \subseteq b \rightsquigarrow$ "a is included in b"

- Labelled formulas

Enriching the language: conditional logics

- Countably many variables for worlds $x, y, z, \dots$
- Countably many variables for neighbourhoods $a, b, c, \dots$

- Relational atoms
  - $x \in a \rightsquigarrow$ "x is an element of a"
  - $a \in N(x) \rightsquigarrow$ "a is an element of $N(x)$"
  - $a \subseteq b \rightsquigarrow$ "a is included in b"

- Labelled formulas
  - $x : A \rightsquigarrow$ "x satisfies A"

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \ldots$

- ☞ Relational atoms
  - ▹ $x \in a$ $\rightsquigarrow$ "x is an element of a"
  - ▹ $a \in N(x)$ $\rightsquigarrow$ "a is an element of $N(x)$"
  - ▹ $a \subseteq b$ $\rightsquigarrow$ "a is included in b"

- ☞ Labelled formulas
  - ▹ $x : A$ $\rightsquigarrow$ "x satisfies A"

$x \Vdash A > B$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} A$, then there is $\beta \in N(x)$ s.t. $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \rightarrow B$

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \ldots$

- ☞ Relational atoms
  - ▹ $x \in a$ ⇝ "x is an element of a"
  - ▹ $a \in N(x)$ ⇝ "a is an element of $N(x)$"
  - ▹ $a \subseteq b$ ⇝ "a is included in b"

- ☞ Labelled formulas
  - ▹ $x : A$ ⇝ "x satisfies A"
  - ▹ $a \Vdash^{\exists} A$ ⇝ "A is satisfied at some world of a"

$x \Vdash A > B$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} A$, then
there is $\beta \in N(x)$ s.t. $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \to B$

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \ldots$

- ☞ Relational atoms
  - ▹ $x \in a \rightsquigarrow$ "x is an element of a"
  - ▹ $a \in N(x) \rightsquigarrow$ "a is an element of $N(x)$"
  - ▹ $a \subseteq b \rightsquigarrow$ "a is included in b"

- ☞ Labelled formulas
  - ▹ $x : A \rightsquigarrow$ "x satisfies A"
  - ▹ $a \Vdash^{\exists} A \rightsquigarrow$ "A is satisfied at some world of a"
  - ▹ $a \Vdash^{\forall} A \rightsquigarrow$ "A is satisfied at all worlds of a"

$x \Vdash A > B$ *iff* for all $\alpha \in N(x)$, if $\alpha \Vdash^{\exists} A$, then there is $\beta \in N(x)$ s.t. $\beta \subseteq \alpha$ and $\beta \Vdash^{\exists} A$ and $\beta \Vdash^{\forall} A \rightarrow B$

Enriching the language: conditional logics

- ☞ Countably many variables for worlds $x, y, z, \ldots$
- ☞ Countably many variables for neighbourhoods $a, b, c, \ldots$

☞ Relational atoms

- ▹ $x \in a \rightsquigarrow$ "x is an element of a"
- ▹ $a \in N(x) \rightsquigarrow$ "a is an element of $N(x)$"
- ▹ $a \subseteq b \rightsquigarrow$ "a is included in b"

☞ Labelled formulas

- ▹ $x : A \rightsquigarrow$ "x satisfies A"
- ▹ $a \Vdash^{\exists} A \rightsquigarrow$ "A is satisfied at some world of a"
- ▹ $a \Vdash^{\forall} A \rightsquigarrow$ "A is satisfied at all worlds of a"
- ▹ $x \Vdash_a A \mid B \rightsquigarrow$ "there is a $b \in N(x)$ such that $b \subseteq a$, $b \Vdash^{\exists} A$ and $b \Vdash^{\forall} A \rightarrow B$"

$$x \Vdash A > B \text{ iff for all } \alpha \in N(x), \text{ if } \alpha \Vdash^{\exists} A, \text{ then}$$
$$\text{there is } \beta \in N(x) \text{ s.t. } \beta \subseteq \alpha \text{ and } \beta \Vdash^{\exists} A \text{ and } \beta \Vdash^{\forall} A \rightarrow B$$

Labelled rules

Labelled rules

☞ Rules for >

$$>_R \frac{a \in N(x), \mathcal{R}, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A > B} \text{ (a!)}$$

$$>_L \frac{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad a \in N(x), \mathcal{R}, x \Vdash_a A \mid B, x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta}$$

$$\Vdash_R \frac{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\exists} A \quad c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\forall} A \to B}{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}$$

$$\Vdash_L \frac{b \in N(x), b \subseteq a, \mathcal{R}, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x \Vdash_a A \mid B, \Gamma \Rightarrow \Delta} \text{ (a!)}$$

Labelled rules

☞ Rules for >

$$>_{\mathrm{R}}\ \frac{a \in N(x), \mathcal{R}, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A > B}\ (a!)$$

$$>_{\mathrm{L}}\ \frac{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad a \in N(x), \mathcal{R}, x \Vdash_a A \mid B, x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta}$$

$$\Vdash_{\mathrm{R}}\ \frac{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\exists} A \quad c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\forall} A \rightarrow B}{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}$$

$$\Vdash_{\mathrm{L}}\ \frac{b \in N(x), b \subseteq a, \mathcal{R}, c \Vdash^{\exists} A, c \Vdash^{\forall} A \rightarrow B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x \Vdash_a A \mid B, \Gamma \Rightarrow \Delta}\ (a!)$$

☞ Rules for frame conditions, example: centering

C For all x, for all $\alpha \in N(x)$, $\{x\} \in N(x)$ and $x \in \alpha$

$$\mathrm{C}\ \frac{\{x\} \in N(x), \{x\} \subseteq a, a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Labelled rules

☞ Rules for >

$$>_R \frac{a \in N(x), \mathcal{R}, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A > B} \text{ (a!)}$$

$$>_L \frac{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad a \in N(x), \mathcal{R}, x \Vdash_a A \mid B, x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta}$$

$$\Vdash_R \frac{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\exists} A \quad c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B, c \Vdash^{\forall} A \to B}{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_a A \mid B}$$

$$\Vdash_L \frac{b \in N(x), b \subseteq a, \mathcal{R}, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x \Vdash_a A \mid B, \Gamma \Rightarrow \Delta} \text{ (a!)}$$

☞ Rules for frame conditions, example: centering

C For all x, for all $\alpha \in N(x)$, $\{x\} \in N(x)$ and $x \in \alpha$

$$\text{C} \frac{\{x\} \in N(x), \{x\} \subseteq a, a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{single} \frac{x \in \{x\}, \{x\} \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{\{x\} \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Labelled rules

☞ Rules for >

$$>_{R}\ \frac{a \in N(x), \mathcal{R}, a \Vdash^{\exists} A, \Gamma \Rightarrow \Delta, x \Vdash_{a} A \mid B}{\mathcal{R}, \Gamma \Rightarrow \Delta, x : A > B}\ (a!)$$

$$>_{L}\ \frac{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta, a \Vdash^{\exists} A \quad a \in N(x), \mathcal{R}, x \Vdash_{a} A \mid B, x : A > B, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, x : A > B, \Gamma \Rightarrow \Delta}$$

$$\Vdash_{R}\ \frac{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_{a} A \mid B, c \Vdash^{\exists} A \quad c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_{a} A \mid B, c \Vdash^{\forall} A \to B}{c \in N(x), c \subseteq a, \mathcal{R}, \Gamma \Rightarrow \Delta, x \Vdash_{a} A \mid B}$$

$$\Vdash_{L}\ \frac{b \in N(x), b \subseteq a, \mathcal{R}, c \Vdash^{\exists} A, c \Vdash^{\forall} A \to B, \Gamma \Rightarrow \Delta}{\mathcal{R}, x \Vdash_{a} A \mid B, \Gamma \Rightarrow \Delta}\ (a!)$$

☞ Rules for frame conditions, example: centering

C For all x, for all $\alpha \in N(x)$, $\{x\} \in N(x)$ and $x \in \alpha$

$$C\ \frac{\{x\} \in N(x), \{x\} \subseteq a, a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{a \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{single}\ \frac{x \in \{x\}, \{x\} \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{\{x\} \in N(x), \mathcal{R}, \Gamma \Rightarrow \Delta}$$

$$\text{Repl}_1\ \frac{y \in \{x\}, At(y), At(x), \mathcal{R}, \Gamma \Rightarrow \Delta}{y \in \{x\}, At(x), \mathcal{R}, \Gamma \Rightarrow \Delta} \qquad \text{Repl}_2\ \frac{y \in \{x\}, At(x), At(y), \mathcal{R}, \Gamma \Rightarrow \Delta}{y \in \{x\}, At(y), \mathcal{R}, \Gamma \Rightarrow \Delta}$$

Example

Axiom c $(A \wedge B) \rightarrow (A > B)$

$$
{\scriptstyle \rightarrow_{\mathrm{R}}}\dfrac{
{\scriptstyle \wedge_{\mathrm{L}}}\dfrac{
{\scriptstyle >_{\mathrm{R}}}\dfrac{
{\scriptstyle \mathrm{C}}\dfrac{
{\scriptstyle \Vdash_{\mathrm{L}}}\dfrac{
{\scriptstyle \mathrm{Single}}\dfrac{
{\scriptstyle \Vdash^{\exists}_{\mathrm{R}}}\dfrac{\ldots x \in \{x\}, x : p \Rightarrow \{x\} \Vdash^{\exists} p, x : p}{\ldots x \in \{x\}, x : p \Rightarrow \{x\} \Vdash^{\exists} p}
}{\ldots x : p \Rightarrow \{x\} \Vdash^{\exists} p}
\quad
{\scriptstyle \Vdash^{\exists}_{\mathrm{R}}}\dfrac{
{\scriptstyle \mathrm{Repl}_1}\dfrac{
{\scriptstyle \rightarrow_{\mathrm{R}}}\dfrac{y \in \{x\}, \ldots, y : q, y : p \Rightarrow y : q}{y \in \{x\}, \ldots, y : q \Rightarrow y : p \rightarrow q}
}{y \in \{x\}, \ldots, x : q \Rightarrow y : p \rightarrow q}
}{\ldots x : q \Rightarrow \{x\} \Vdash^{\forall} p \rightarrow q}
}{\{x\} \in a, \{x\} \subseteq a, a \in N(x), a \Vdash^{\exists} A, x : p, x : q \Rightarrow x \Vdash_a p \mid q}
}{a \in N(x), a \Vdash^{\exists} A, x : p, x : q \Rightarrow x \Vdash_a p \mid q}
}{x : p, x : q \Rightarrow x : p > q}
}{x : p \wedge q \Rightarrow x : p > q}
}{\Rightarrow x : (p \wedge q) \rightarrow x : (p > q)}
$$

Main results [G, Negri and Olivetti, 2021]

Main results [G, Negri and Olivetti, 2021]

☞ For L any logic in the conditional lattice

Theorem (Completeness, I). If A is derivable from the axioms for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Main results [G, Negri and Olivetti, 2021]

☞ For L any logic in the conditional lattice

Theorem (Completeness, I). If A is derivable from the axioms for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Proof. By proving cut-admissibility (easy).

☞ For L any logic in the conditional lattice

Theorem (Completeness, I). If A is derivable from the axioms for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Proof. By proving cut-admissibility (easy).

☞ For L any logic in the conditional lattice **without** absoluteness

Theorem (Completeness, II). If A is valid in the class of models for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

☞ For L any logic in the conditional lattice

Theorem (Completeness, I). If A is derivable from the axioms for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Proof. By proving cut-admissibility (easy).

☞ For L any logic in the conditional lattice **without** absoluteness

Theorem (Completeness, II). If A is valid in the class of models for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Proof. Show that if A is not provable, we can construct a finite countermodel for it (easy).

Main results [G, Negri and Olivetti, 2021]

☞ For L any logic in the conditional lattice

Theorem (Completeness, I). If A is derivable from the axioms for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

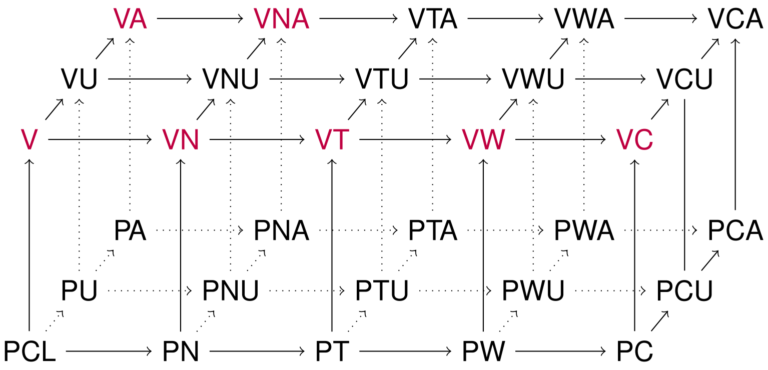
Proof. By proving cut-admissibility (easy).

☞ For L any logic in the conditional lattice **without** absoluteness

Theorem (Completeness, II). If A is valid in the class of models for L, then $\Rightarrow x : A$ is provable in the labelled calculus for L.

Proof. Show that if A is not provable, we can construct a finite countermodel for it (easy). We need to show termination (difficult).

Sequent calculi with blocks for (some) Lewis' logics



Sequents with blocks

Blocks (Σ multiset of formulas) [Olivetti & Pozzato, 2015]

$$[\Sigma \lhd C] \quad \rightsquigarrow \quad \bigvee_{B \in \Sigma} (B \preccurlyeq C)$$

Sequents with blocks

Blocks (Σ multiset of formulas) [Olivetti & Pozzato, 2015]

$$[\Sigma \lhd C] \quad \rightsquigarrow \quad \bigvee_{B \in \Sigma} (B \preccurlyeq C)$$

Example $[A, B \lhd C] \quad \rightsquigarrow \quad (A \preccurlyeq C) \vee (B \preccurlyeq C)$

Sequents with blocks

Blocks (Σ multiset of formulas) [Olivetti & Pozzato, 2015]

$$[\Sigma \lhd C] \quad \rightsquigarrow \quad \bigvee_{B \in \Sigma} (B \preccurlyeq C)$$

Example $[A, B \lhd C] \quad \rightsquigarrow \quad (A \preccurlyeq C) \vee (B \preccurlyeq C)$

Sequents with blocks (Γ, Δ multisets of formulas)

Sequents with blocks

Blocks (Σ multiset of formulas) [Olivetti & Pozzato, 2015]

$$[\Sigma \lhd C] \quad \rightsquigarrow \quad \bigvee_{B \in \Sigma} (B \preccurlyeq C)$$

Example $[A, B \lhd C] \quad \rightsquigarrow \quad (A \preccurlyeq C) \vee (B \preccurlyeq C)$

Sequents with blocks (Γ, Δ multisets of formulas)

$$\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd C_1], \ldots, [\Sigma_k \lhd C_k]$$

Sequents with blocks

Blocks (Σ multiset of formulas) [Olivetti & Pozzato, 2015]

$$[\Sigma \lhd C] \quad \rightsquigarrow \quad \bigvee_{B \in \Sigma} (B \preccurlyeq C)$$

Example $[A, B \lhd C] \quad \rightsquigarrow \quad (A \preccurlyeq C) \vee (B \preccurlyeq C)$

Sequents with blocks (Γ, Δ multisets of formulas)

$$\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd C_1], \ldots, [\Sigma_k \lhd C_k] \quad \rightsquigarrow$$

$$\bigwedge \Gamma \rightarrow \bigvee \Delta \vee (\bigvee_{B \in \Sigma_1} (B \preccurlyeq C_1)) \vee \cdots \vee (\bigvee_{B \in \Sigma_k} (B \preccurlyeq C_k))$$

The rules

☞ Rules for V

$$\text{init}\,\frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_{\mathrm{L}}\,\frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_{\mathrm{R}}\,\frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_{\mathrm{L}}\,\frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

The rules

☞ Rules for V

$$\text{init}\;\frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_L\;\frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_R\;\frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_L\;\frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

$$\preccurlyeq_R\;\frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B}$$

The rules

☞ Rules for V

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_R\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_L\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

$$\preccurlyeq_R\ \frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B}$$

$$\preccurlyeq_L\ \frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C], [\Sigma \lhd A]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C]}$$

The rules

☞ Rules for V

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \rightarrow_R\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \qquad \rightarrow_L\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \rightarrow B \Rightarrow \Delta}$$

$$\preccurlyeq_R\ \frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B}$$

$$\preccurlyeq_L\ \frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C], [\Sigma \lhd A]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C]}$$

$$\text{com}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}$$

The rules

☞ Rules for V

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_R\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_L\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

$$\preccurlyeq_R\ \frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B} \qquad \text{jump}\ \frac{B \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd B]}$$

$$\preccurlyeq_L\ \frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C], [\Sigma \lhd A]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C]}$$

$$\text{com}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}$$

The rules

☞ Rules for V

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \to_R\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \to B} \qquad \to_L\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \to B \Rightarrow \Delta}$$

$$\preccurlyeq_R\ \frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B} \qquad \text{jump}\ \frac{B \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd B]}$$

$$\preccurlyeq_L\ \frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C], [\Sigma \lhd A]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C]}$$

$$\text{com}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}$$

$$>_L\ \frac{\bot \preccurlyeq A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, [A \wedge \neg B \lhd A]}{A > B, \Gamma \Rightarrow \Delta} \qquad >_R\ \frac{(A \wedge \neg B) \preccurlyeq A, \Gamma \Rightarrow \Delta, [\bot \lhd A]}{\Gamma \Rightarrow \Delta, A > B}$$

$$A > B \;:=\; (\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \vee B))$$

The rules

☞ Rules for V

$$\text{init}\ \frac{}{\Gamma, p \Rightarrow p, \Delta} \qquad \bot_L\ \frac{}{\Gamma, \bot \Rightarrow \Delta} \qquad \rightarrow_R\ \frac{\Gamma, A \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \rightarrow B} \qquad \rightarrow_L\ \frac{\Gamma, B \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, A}{\Gamma, A \rightarrow B \Rightarrow \Delta}$$

$$\preccurlyeq_R\ \frac{\Gamma \Rightarrow \Delta, [A \lhd B]}{\Gamma \Rightarrow \Delta, A \preccurlyeq B} \qquad \text{jump}\ \frac{B \Rightarrow \Sigma}{\Gamma \Rightarrow \Delta, [\Sigma \lhd B]}$$

$$\preccurlyeq_L\ \frac{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [B, \Sigma \lhd C] \quad \Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C], [\Sigma \lhd A]}{\Gamma, A \preccurlyeq B \Rightarrow \Delta, [\Sigma \lhd C]}$$

$$\text{com}\ \frac{\Gamma \Rightarrow \Delta, [\Sigma_1, \Sigma_2 \lhd A], [\Sigma_2 \lhd B] \quad \Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_1, \Sigma_2 \lhd B]}{\Gamma \Rightarrow \Delta, [\Sigma_1 \lhd A], [\Sigma_2 \lhd B]}$$

$$>_L\ \frac{\bot \preccurlyeq A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, [A \wedge \neg B \lhd A]}{A > B, \Gamma \Rightarrow \Delta} \qquad >_R\ \frac{(A \wedge \neg B) \preccurlyeq A, \Gamma \Rightarrow \Delta, [\bot \lhd A]}{\Gamma \Rightarrow \Delta, A > B}$$

$$A > B \ := \ (\bot \preccurlyeq A) \vee \neg((A \wedge \neg B) \preccurlyeq (A \vee B))$$

☞ Rules for extensions, example: centering

$$\text{C}\ \frac{A, \Gamma \Rightarrow \Delta \quad \Gamma \Rightarrow \Delta, B}{A \preccurlyeq B, \Gamma \Rightarrow \Delta}$$

Examples

Axiom $(A \preccurlyeq B) \vee (B \preccurlyeq A)$

$$
\vee_R\,\dfrac{\preccurlyeq_R\,\dfrac{\preccurlyeq_R\,\dfrac{\text{com}\,\dfrac{\text{jump}\,\dfrac{\text{init}\,\dfrac{}{b \Rightarrow a, b}}{\Rightarrow a \preccurlyeq b, b \preccurlyeq a, [a, b \lhd b], [b \lhd a]} \qquad \text{jump}\,\dfrac{\text{init}\,\dfrac{}{a \Rightarrow a, b}}{\Rightarrow a \preccurlyeq b, b \preccurlyeq a, [a \lhd b], [a, b \lhd a]}}{\Rightarrow a \preccurlyeq b, b \preccurlyeq a, [a \lhd b], [b \lhd a]}}{\Rightarrow a \preccurlyeq b, b \preccurlyeq a, [a \lhd b]}}{\Rightarrow a \preccurlyeq b, b \preccurlyeq a}}{\Rightarrow (a \preccurlyeq b) \vee (b \preccurlyeq a)}
$$

Axiom c $(A \vee B) \rightarrow (A > B)$

$$
\rightarrow_R\,\dfrac{\wedge_L\,\dfrac{>_R\,\dfrac{\text{C}\,\dfrac{\wedge_L\,\dfrac{\neg_L\,\dfrac{p, p, q \Rightarrow [\bot \lhd p], q}{p, \neg q, p, q \Rightarrow [\bot \lhd p]}}{p \wedge \neg q, p, q \Rightarrow [\bot \lhd p]} \qquad p, q \Rightarrow [\bot \lhd p], p}{(p \wedge \neg q) \preccurlyeq p, p, q \Rightarrow [\bot \lhd p]}}{p, q \Rightarrow p > q}}{p \wedge q \Rightarrow p > q}}{\Rightarrow (p \wedge q) \rightarrow (p > q)}
$$

Main results [G Lellmann, Olivetti, Pozzato, 2016]

Main results [G Lellmann, Olivetti, Pozzato, 2016]

☞ For L any logic in V, VN, VT, VW, VC, VA, VNA

Theorem (Completeness, I). If A is derivable from the axioms for L, then A is provable in the sequent calculus w. blocks for L.

Main results [G Lellmann, Olivetti, Pozzato, 2016]

☞ For L any logic in V, VN, VT, VW, VC, VA, VNA

Theorem (Completeness, I). If A is derivable from the axioms for L, then A is provable in the sequent calculus w. blocks for L.

Proof. For V, by proving cut-admissibility (difficult). For the other logics, by simulating cut-free proofs of a non-standard calculus.

Main results [G Lellmann, Olivetti, Pozzato, 2016]

☞ For L any logic in V, VN, VT, VW, VC, VA, VNA

Theorem (Completeness, I). If A is derivable from the axioms for L, then A is provable in the sequent calculus w. blocks for L.

Proof. For V, by proving cut-admissibility (difficult). For the other logics, by simulating cut-free proofs of a non-standard calculus.

Theorem (Completeness, II). If A is valid in the class of models for L, then A is provable in the labelled calculus for L.

Main results [G Lellmann, Olivetti, Pozzato, 2016]

☞ For L any logic in V, VN, VT, VW, VC, VA, VNA

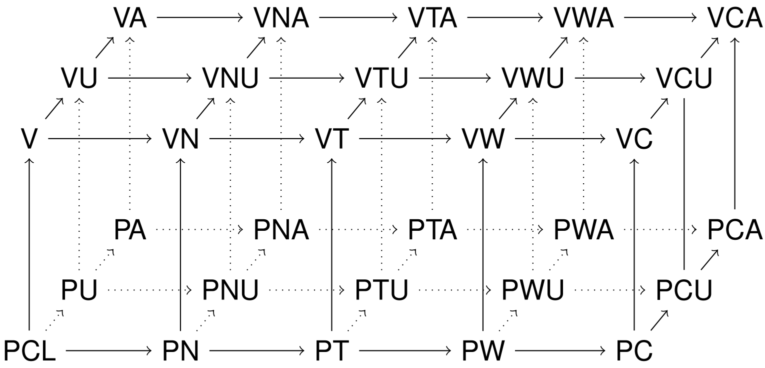
Theorem (Completeness, I). If A is derivable from the axioms for L, then A is provable in the sequent calculus w. blocks for L.

Proof. For V, by proving cut-admissibility (difficult). For the other logics, by simulating cut-free proofs of a non-standard calculus.

Theorem (Completeness, II). If A is valid in the class of models for L, then A is provable in the labelled calculus for L.

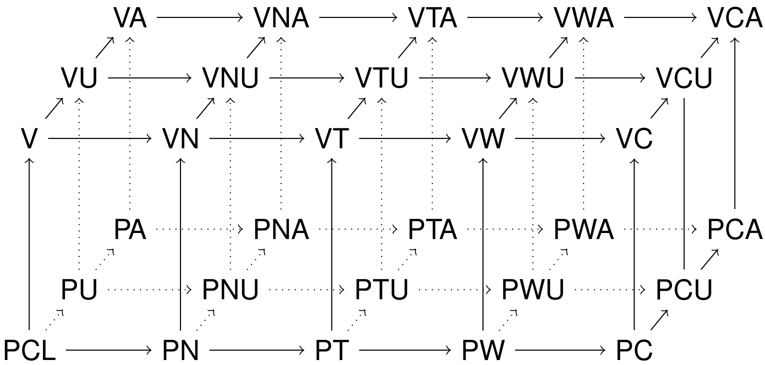
Proof. Show that if A is not provable, we can construct a finite countermodel for it (difficult). We need to show termination (easy).

Summing up



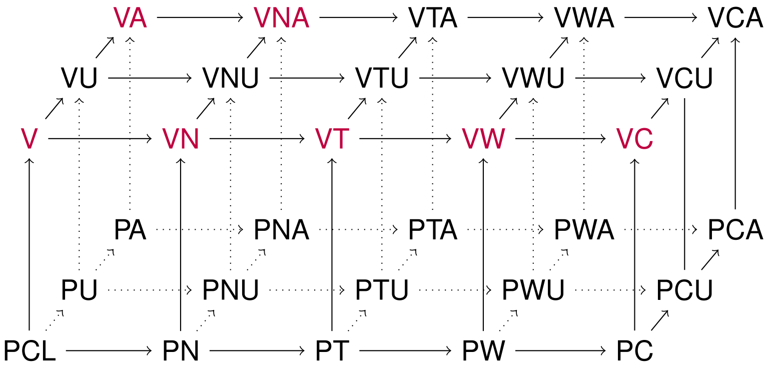
VA
VNA
VTA
VWA
VCA
VU
VNU
VTU
VWU
VCU
V
VN
VT
VW
VC
PA
PNA
PTA
PWA
PCA
PU
PNU
PTU
PWU
PCU
PCL
PN
PT
PW
PC

Summing up



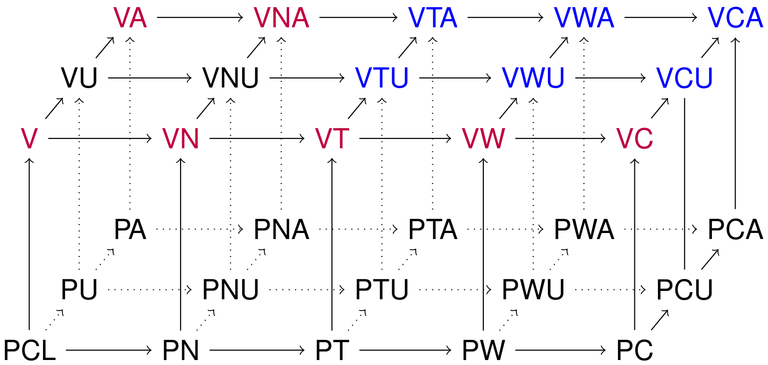
- Labelled calculi for all the logics [G, Negri, Olivetti, 2021]

Summing up



- Labelled calculi for all the logics [G, Negri, Olivetti, 2021]
- Sequent calculus with blocks for V, VN, VT, VW, VC, VA, VNA [G, Lellmann, Olivetti, Pozzato, 2016]

Summing up



- Labelled calculi for all the logics [G, Negri, Olivetti, 2021]
- Sequent calculus with blocks for V, VN, VT, VW, VC, VA, VNA [G, Lellmann, Olivetti, Pozzato, 2016]
- Hypersequent calculus with blocks for logics VTU, VWU, VCU, VTA, VWA, VCA [G, Lellmann, Olivetti, Pozzato, 2017]

Conclusions

	formula interpretation	direct cut adm.	termination of proof search	countermodel construction
labelled	no	easy	difficult	easy
blocks	yes	difficult	easy	difficult

Conclusions

	formula interpretation	direct cut adm.	termination of proof search	countermodel construction
labelled	no	easy	difficult	easy
blocks	yes	difficult	easy	difficult

Current & Future work:

Conclusions

	formula interpretation	direct cut adm.	termination of proof search	countermodel construction
labelled	no	easy	difficult	easy
blocks	yes	difficult	easy	difficult

Current & Future work:

- Explore the proof theory of logics with the comparative plausibility operator, **without** nesting [Dalmonte, G, 2022]

Conclusions

	formula interpretation	direct cut adm.	termination of proof search	countermodel construction
labelled	no	easy	difficult	easy
blocks	yes	difficult	easy	difficult

Current & Future work:

- Explore the proof theory of logics with the comparative plausibility operator, **without** nesting [Dalmonte, G, 2022]
- Evaluating *ceteris paribus* counterfacuals [Delkos, G, Arxiv]

Conclusions

	formula interpretation	direct cut adm.	termination of proof search	countermodel construction
labelled	no	easy	difficult	easy
blocks	yes	difficult	easy	difficult

Current & Future work:

- Explore the proof theory of logics with the comparative plausibility operator, **without** nesting [Dalmonte, G, 2022]
- Evaluating *ceteris paribus* counterfacuals [Delkos, G, Arxiv]
- Explore the proof theory of intuitionistic conditional logics [Ciardelli, Liu, 2019]

Conclusions

	formula interpretation	direct cut adm.	termination of proof search	countermodel construction
labelled	no	easy	difficult	easy
blocks	yes	difficult	easy	difficult

Current & Future work:

- Explore the proof theory of logics with the comparative plausibility operator, **without** nesting [Dalmonte, G, 2022]
- Evaluating *ceteris paribus* counterfacuals [Delkos, G, Arxiv]
- Explore the proof theory of intuitionistic conditional logics [Ciardelli, Liu, 2019]

Thank you!