Inquisitive logic with modal operators that quantify over alternatives

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Quantifying over alternatives

The problem of free choice permission is to account for the apparent validity of inferences like:

If you may take an apple or a pear, you may take an apple and you may take a pear.

One way to capture this kind of free-choice effect is to interpret disjunction as introducing sets of alternative propositions corresponding to the disjuncts, and interpret 'may' as quantifying over the alternatives in this set (e.g. [Aloni 2007, Aloni & Ciardelli 2013]).

'$May(p \text{ or } q)$' is true	iff	each alternative in $\textbf{alt}(p \text{ or } q) = \{|p|, |q|\}$ contains some ideal world
	iff	$|p|$ contains some ideal world, and $|q|$ contains some ideal world
	iff	'$May(p)$' is true and '$May(q)$' is true

The topic of this talk: Introduce and axiomatize an extension of inquisitive logic with a modal operator interpreted along these lines.

Background on propositional inquisitive logic

Inquisitive logic

- Inquisitive logic captures the logical relations between both statements and questions (see e.g. [Ciardelli 2016, Ciardelli 2022, Ciardelli, Groenendijk & Roelofsen 2018]).
- Formulas are evaluated in terms of support at information states:
    - a statement is supported by an information state if the information contained in the state implies that the statement is true;
    - a question is supported by an information state if the information contained in the state settles the issue raised by the question.
- The maximal information states supporting a formula are called the alternatives for the formula.
- The topic of this talk: Extensions of propositional inquisitive logic featuring modal operators with semantics defined in terms of quantification over the alternatives for their argument formulas.

Language of propositional inquisitive logic

Language for propositional inquisitive logic

Language $\mathcal{L}$, where p ranges over a countable set **Prop** of atomic proposition symbols:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \mathbin{\mathpalette\make@circled\vee} \varphi$$

Abbreviations:

$$\neg\varphi := \varphi \to \bot, \qquad \varphi \vee \psi := \neg(\neg\varphi \wedge \neg\psi), \qquad \varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$$

Inquisitive disjunction $\mathbin{\vee\!\!\!\vee}$ is used to form questions:

$p \mathbin{\vee\!\!\!\vee} \neg p$	whether p	*Is it raining?*
$p \mathbin{\vee\!\!\!\vee} q$	whether p or q	*Is it raining or snowing?*
$(p \mathbin{\vee\!\!\!\vee} \neg p) \wedge (q \mathbin{\vee\!\!\!\vee} \neg q)$	whether p and whether q	*Is it raining, and is it snowing?*
$p \to (q \mathbin{\vee\!\!\!\vee} \neg q)$	whether q, if p	*If it is raining, is it snowing?*

Models, information states and support

A model is a pair $\mathcal{M} = (W, V)$ where

- W is a set of (of possible worlds),
- $V : \textbf{Prop} \to \mathcal{P}(W)$ is a function interpreting atomic propositions.

Any subset $X \subseteq W$ is an information state.

Semantics in terms of support at information states

$$
\begin{array}{lll}
\mathcal{M}, X \models p & \text{iff} & X \subseteq V(p) \\
\mathcal{M}, X \models \bot & \text{iff} & X = \emptyset \\
\mathcal{M}, X \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, X \models \varphi \text{ and } \mathcal{M}, X \models \psi \\
\mathcal{M}, X \models \varphi \to \psi & \text{iff} & \text{for all } Y \subseteq X,\ \mathcal{M}, Y \models \varphi \text{ implies } \mathcal{M}, Y \models \psi \\
\mathcal{M}, X \models \varphi \mathbin{\mathbb{V}} \psi & \text{iff} & \mathcal{M}, X \models \varphi \text{ or } \mathcal{M}, X \models \psi
\end{array}
$$

Truth, truth-conditionality, validity, logical consequence

Truth: $\mathcal{M}, w \models \varphi$ iff $\mathcal{M}, \{w\} \models \varphi$.

Truth set of a formula: $|\varphi|_{\mathcal{M}} = \{w \in W \mid \mathcal{M}, w \models \varphi\}$

Truth-conditionality: For all $\mathcal{M}$, for all X in $\mathcal{M}$: $\mathcal{M}, X \models \varphi$ iff for all $w \in X$, $\mathcal{M}, w \models \varphi$.

Validity in a model: $\mathcal{M} \models \varphi$ iff for all X in $\mathcal{M}$: $\mathcal{M}, X \models \varphi$

Validity: $\models \varphi$ iff for all $\mathcal{M}$: $\mathcal{M} \models \varphi$

Logical consequence: $\Phi \models \psi$ iff for all $\mathcal{M}$ and all X in $\mathcal{M}$: if $\mathcal{M}, X \models \Phi$ then $\mathcal{M}, X \models \psi$.

Alternatives

The alternatives for φ are the $\subseteq$-maximal information states supporting φ:

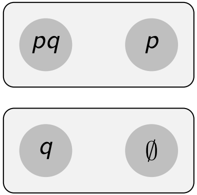
$$\mathbf{alt}_{\mathcal{M}}(\varphi) = \{X \subseteq W \mid \mathcal{M}, X \models \varphi \text{ and for all } Y \text{ s.t. } \mathcal{M}, Y \models \varphi, \text{ if } X \subseteq Y \text{ then } X = Y\}$$

Proposition (Normality)

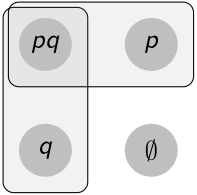
For any formula φ, any model $\mathcal{M}$ and any info state X:

$$\mathcal{M}, X \models \varphi \quad \textit{iff} \quad X \subseteq Y \textit{ for some } Y \in \mathbf{alt}_{\mathcal{M}}(\varphi).$$

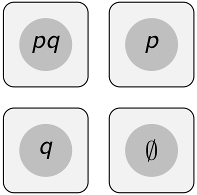
Alternatives: some examples



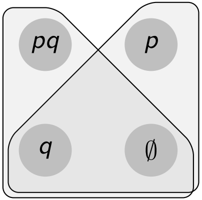
$p \mathbin{\mathbb{V}} \neg p$



$p \mathbin{\mathbb{V}} q$



$(p \mathbin{\mathbb{V}} \neg p) \wedge (q \mathbin{\mathbb{V}} \neg q)$



$p \rightarrow (q \mathbin{\mathbb{V}} \neg q)$

A modal operator that quantifies over alternatives

Adding a modal operator that quantifies over alternatives

In [Nygren 2023], I extend propositional inquisitive logic with a modal operator $\Diamond$ that generalizes the standard existential modal operator to the inquisitive logic setting.

A formula of the form $\Diamond\varphi$ expresses that each alternative for φ is true at some accessible world.

For example, provided that the model $\mathcal{M}$ is such that $|p|_{\mathcal{M}}, |q|_{\mathcal{M}} \in \mathbf{alt}_{\mathcal{M}}(p \vee\!\!\!\vee q)$, $\Diamond(p \vee\!\!\!\vee q)$ expresses that p is true at some accessible world, and that q is true at some accessible world.

Language and semantics

Language extended with $\Diamond$

Language $\mathcal{L}_\Diamond$, where p ranges over a countable set **Prop** of atomic proposition symbols:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \vee\!\!\!\vee \varphi \mid \Diamond\varphi$$

The language is interpreted on Kripke models $\mathcal{M} = (W, R, V)$.

New support clause:

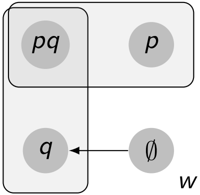
$$\mathcal{M}, X \models \Diamond\varphi \quad \text{iff} \quad \text{for all } w \in X, \text{ for all } Y \in \mathbf{alt}_{\mathcal{M}}(\varphi),\ Y \cap R[w] \neq \emptyset.$$

Here, $\mathbf{alt}_{\mathcal{M}}(\varphi)$ is defined as before.

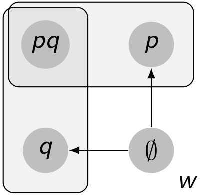
Note: This support clause only works as intended since each formula has an associated set of alternatives, i.e. since each formula is normal.

The notions of truth, truth-conditionality, truth sets, validity and logical consequence are defined in the same way as before.

Examples



$w \not\models \Diamond(p \vee q)$



$w \models \Diamond(p \vee q)$

Free choice

Provided that p and q are independent in the model $\mathcal{M}$ in the sense that neither of $|p|_{\mathcal{M}}$ and $|q|_{\mathcal{M}}$ is included in the other, the following hold:

$$\mathcal{M} \models \Diamond(p \varovee q) \to \Diamond p \wedge \Diamond q.$$

Thus, $\Diamond$ can be used to capture (at least some) properties of free-choice inferences.

(However, the account cannot handle e.g. dual prohibition or wide scope FC – see recent work on BSML (e.g. [Aloni 2022, Aloni, Anttila & Yang 2023]) for an alternative!)

Ignorance

Under an epistemic interpretation, $\Diamond\varphi$ means that an implicit agent's information state is consistent with all alternatives for φ.

The $\Diamond$ modality can be combined with a generalized box modality to capture a notion of ignorance with respect to a question.

Generalized box modality [Ciardelli 2016]:

$$\mathcal{M}, X \models \Box\varphi \quad \text{iff} \quad \text{for all } w \in X,\ \mathcal{M}, R[w] \models \varphi.$$

Epistemic interpretation: $\Box\varphi$ means that the agent's information state supports φ, or in other words that the agent knows φ.

Define $\mathbf{I}\varphi := \neg\Box\varphi \wedge \Diamond\varphi$.

When φ is a question, $\mathbf{I}\varphi$ says that the agent does not know an answer to φ, and, moreover, is not able to rule out any of the alternatives for φ as being incorrect.

In other words, $\mathbf{I}\varphi$ says that the agent is completely ignorant with respect to φ.

The account generalizes previous work on ignorance whether [van der Hoek & Lomuscio 2004].

Some properties

When α is truth-conditional, $\Diamond\alpha$ behaves just like an ordinary existential modal operator:

$$\mathcal{M}, w \models \Diamond\alpha \quad \text{iff} \quad \text{there is } v \in R[w] \text{ such that } \mathcal{M}, v \models \alpha.$$

What about the interaction with inquisitive disjunction?

Since

$$\mathbf{alt}_{\mathcal{M}}(\varphi \vee\!\!\!\vee \psi) \subseteq \mathbf{alt}_{\mathcal{M}}(\varphi) \cup \mathbf{alt}_{\mathcal{M}}(\psi),$$

the following validity holds:

$$\models \Diamond\varphi \wedge \Diamond\psi \rightarrow \Diamond(\varphi \vee\!\!\!\vee \psi).$$

However, since it is not in general the case that

$$\mathbf{alt}_{\mathcal{M}}(\varphi) \cup \mathbf{alt}_{\mathcal{M}}(\psi) \subseteq \mathbf{alt}_{\mathcal{M}}(\varphi \vee\!\!\!\vee \psi),$$

we have:

$$\not\models \Diamond(\varphi \vee\!\!\!\vee \psi) \rightarrow \Diamond\varphi, \qquad \not\models \Diamond(\varphi \vee\!\!\!\vee \psi) \rightarrow \Diamond\psi$$

Adding the global modality

One way to handle disjunctions in the scope of $\Diamond$ is to add an additional global modal operator that allows talking about the alternatives for formulas in the object language.

Language extended with $\Diamond$ and $\mathbf{A}$

Language $\mathcal{L}_{\Diamond\mathbf{A}}$, where p ranges over **Prop**:

$$\varphi ::= p \mid \bot \mid \varphi \wedge \varphi \mid \varphi \to \varphi \mid \varphi \vee\!\!\!\vee\, \varphi \mid \Diamond\varphi \mid \mathbf{A}\varphi$$

A is a variant of the global modality [Goranko & Passy 1992], adapted to the inquisitive logic setting.

Where $\mathcal{M} = (W, R, V)$ is a Kripke model:

$$\mathcal{M}, X \models \mathbf{A}\varphi \quad \text{iff} \quad \mathcal{M}, W \models \varphi.$$

That is: $\mathbf{A}\varphi$ is supported iff φ is supported by the maximal info state iff φ is valid in the model.

Using the global modality

Using the global modality, disjunctions in the scope of $\Diamond$ can be simplified.

Lemma

For any model $\mathcal{M}$, any world w of $\mathcal{M}$, any formula φ and any truth-conditional formula α:

$$\mathcal{M}, w \models \neg\mathbf{A}(\alpha \to \varphi) \quad \textit{implies} \quad |\alpha|_{\mathcal{M}} \in \mathbf{alt}_{\mathcal{M}}(\alpha \vee\!\!\!\vee \varphi).$$

Hence, where α is a truth-conditional formula:

$$\models \Diamond(\alpha \vee\!\!\!\vee \varphi) \wedge \neg\mathbf{A}(\alpha \to \varphi) \to \Diamond\alpha.$$

Axiomatization and completeness

General semantics

Definition (General models)

A general model is a structure $\mathcal{M} = (W, S, R, V)$ where W is a set, $S \subseteq W \times W$ is an equivalence relation, $R \subseteq S$ is a relation, and V is a valuation function.

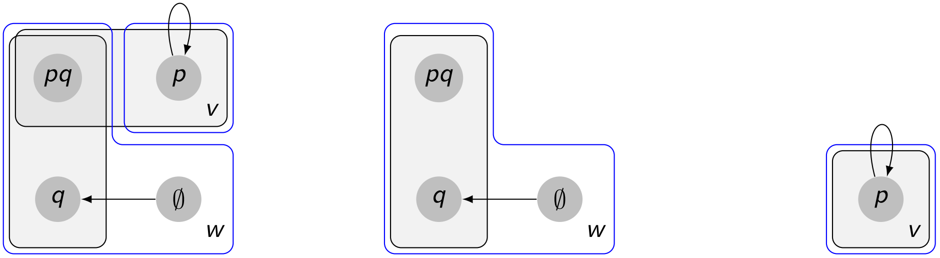
Definition (Generated submodel)

Let $\mathcal{M} = (W, S, R, V)$ be a general model and let $w \in W$. The w-generated submodel $\mathcal{M}[w] = (W_w, S_w, R_w, V_w)$ is the restriction of $\mathcal{M}$ to $S[w]$.

Definition (Support conditions, general semantics)

$$
\begin{array}{lll}
\mathcal{M}, X \models_{\mathbf{G}} \mathbf{A}\varphi & \text{iff} & \text{for all } w \in X,\ \mathcal{M}, S[w] \models_{\mathbf{G}} \varphi; \\
\mathcal{M}, X \models_{\mathbf{G}} \Diamond\varphi & \text{iff} & \text{for all } w \in X,\ \text{for all } Y \in \mathbf{alt}_{\mathcal{M}[w]}(\varphi),\ Y \cap R[w] \neq \emptyset.
\end{array}
$$

Examples



$\mathcal{M}$ $\mathcal{M}[w]$ $\mathcal{M}[v]$

$\mathcal{M}, \{w, v\} \models_{\mathbf{G}} \Diamond(p \vee q)$, since:

- for each $Y \in \mathbf{alt}_{\mathcal{M}[w]}(p \vee q)$, $Y \cap R[w] \neq \emptyset$;
- for each $Y \in \mathbf{alt}_{\mathcal{M}[v]}(p \vee q)$, $Y \cap R[v] \neq \emptyset$.

General semantics

Every standard model can be seen as a special kind of general model, where the equivalence relation S is the universal relation:

Proposition

Let $\mathcal{M} = (W, S, R, V)$ be a general model such that $S = W \times W$. Let $\mathcal{M}'$ be the standard model $\mathcal{M}' = (W, R, V)$. Then for all formulas φ and all info states X: $\mathcal{M}, X \models_{\mathbf{G}} \varphi$ iff $\mathcal{M}', X \models \varphi$.

Declarative formulas

The declarative fragment of the language consists of formulas where $\mathbb{V}$ is only allowed within the scope of a modal operator:

Declarative fragment

Language $\mathcal{L}^d_{\Diamond\mathbf{A}}$, where p ranges over **Prop** and φ ranges over $\mathcal{L}_{\Diamond\mathbf{A}}$:

$$\alpha ::= p \mid \bot \mid \alpha \wedge \alpha \mid \alpha \to \alpha \mid \Diamond\varphi \mid \mathbf{A}\varphi$$

Proposition ([Ciardelli 2016])

All declarative formulas in $\mathcal{L}^d_{\Diamond\mathbf{A}}$ are truth-conditional.

Resolutions and normal form

Any formula of $\mathcal{L}_{\Diamond\mathbf{A}}$ is equivalent to an inquisitive disjunction of declarative formulas.

Definition (Resolutions [Ciardelli 2016])

- $\mathcal{R}(\varphi) = \{\varphi\}$, if $\varphi \in$ **Prop**, $\varphi = \bot$, $\varphi = \Diamond\psi$ or $\varphi = \mathbf{A}\psi$;
- $\mathcal{R}(\varphi \wedge \psi) = \{\alpha \wedge \beta \mid \alpha \in \mathcal{R}(\varphi), \beta \in \mathcal{R}(\psi)\}$;
- $\mathcal{R}(\varphi \to \psi) = \{\bigwedge_{\alpha \in \mathcal{R}(\varphi)}(\alpha \to f(\alpha)) \mid f$ is a function from $\mathcal{R}(\varphi)$ to $\mathcal{R}(\psi)\}$;
- $\mathcal{R}(\varphi \vee\!\!\!\vee \psi) = \mathcal{R}(\varphi) \cup \mathcal{R}(\psi)$.

Proposition (Normal form [Ciardelli 2016])

Any formula $\varphi \in \mathcal{L}_{\Diamond\mathbf{A}}$ is equivalent to $\alpha_1 \vee\!\!\!\vee \cdots \vee\!\!\!\vee \alpha_n$, where $\{\alpha_1, \ldots, \alpha_n\} = \mathcal{R}(\varphi)$.

Filtered resolutions

Let $\mathcal{M}$ be a model. Define the preorder $\leq_{\mathcal{M}}$ ($\mathcal{M}$-entailment) and the equivalence relation $\sim_{\mathcal{M}}$ ($\mathcal{M}$-equivalence) on the set of declarative formulas as follows:

$$\begin{array}{lll} \alpha \leq_{\mathcal{M}} \beta & \text{iff} & |\alpha|_{\mathcal{M}} \subseteq |\beta|_{\mathcal{M}}; \\ \alpha \sim_{\mathcal{M}} \beta & \text{iff} & \alpha \leq_{\mathcal{M}} \beta \text{ and } \beta \leq_{\mathcal{M}} \alpha \text{ (i.e. } |\alpha|_{\mathcal{M}} = |\beta|_{\mathcal{M}}). \end{array}$$

The $\mathcal{M}$-filtered set of resolutions of φ is the set

$$\mathcal{R}_{\mathcal{M}}(\varphi) := \{\alpha \in \mathcal{R}(\varphi) \mid \text{ for all } \beta \in \mathcal{R}(\varphi), \alpha \leq_{\mathcal{M}} \beta \text{ implies } \alpha \sim_{\mathcal{M}} \beta\}.$$

Lemma

Let $\mathcal{M}$ be a general model and φ a formula. Then $\textbf{alt}_{\mathcal{M}}(\varphi) = \{|\alpha|_{\mathcal{M}} \mid \alpha \in \mathcal{R}_{\mathcal{M}}(\varphi)\}$.

Semantic characterization of $\Diamond$

Lemma

Let $\mathcal{M}$ be a general model, $\mathcal{M}[w]$ the w-generated submodel of $\mathcal{M}$ for some world w and let φ be a formula. Then

$$\mathcal{M}, w \models_{\mathbf{G}} \Diamond\varphi \quad \textit{iff} \quad \mathcal{M}, w \models_{\mathbf{G}} \bigwedge_{\alpha \in \mathcal{R}_{\mathcal{M}[w]}(\varphi)} \Diamond\alpha.$$

Axiomatizing logical consequence in general semantics

1. Axioms for prop. inquisitive logic.
2. Axioms for $\mathbf{A}$, where $\alpha \in \mathcal{L}^d_{\Diamond\mathbf{A}}$ is declarative:
   2.1 $\mathbf{A}(\varphi \vee\!\!\!\vee \psi) \to \mathbf{A}\varphi \vee \mathbf{A}\psi$
   2.2 $\mathbf{A}(\varphi \to \psi) \to (\mathbf{A}\varphi \to \mathbf{A}\psi)$
   2.3 $\mathbf{A}\alpha \to \alpha$
   2.4 $\mathbf{A}\varphi \to \mathbf{A}\mathbf{A}\varphi$
   2.5 $\neg\mathbf{A}\varphi \to \mathbf{A}\neg\mathbf{A}\varphi$
3. Axioms for $\Diamond$, where $\alpha, \beta \in \mathcal{L}^d_{\Diamond\mathbf{A}}$ are declaratives:
   3.1 $\Diamond\alpha \to \neg\mathbf{A}\neg\alpha$
   3.2 $\Diamond(\alpha \vee \beta) \leftrightarrow \Diamond\alpha \vee \Diamond\beta$
   3.3 $\mathbf{A}(\varphi \leftrightarrow \psi) \to (\Diamond\varphi \leftrightarrow \Diamond\psi)$
   3.4 $\Diamond\varphi \wedge \Diamond\psi \to \Diamond(\varphi \vee\!\!\!\vee \psi)$
   3.5 $\Diamond(\alpha \vee\!\!\!\vee \varphi) \wedge \neg\mathbf{A}(\alpha \to \varphi) \to \Diamond\alpha$

Rules of inference:

- MP: from φ and $\varphi \to \psi$, infer ψ
- Nec for $\mathbf{A}$: if φ is a theorem, infer $\mathbf{A}\varphi$

Denote derivability in this system by $\vdash_{\mathbf{G}}$. Define $\Phi \vdash_{\mathbf{G}} \psi$ by $\vdash_{\mathbf{G}} \varphi_1 \wedge \cdots \wedge \varphi_n \to \psi$ for some $\varphi_1, \ldots, \varphi_n \in \Phi$.

Complete declarative theories

Definition (Complete declarative theories)

A set $\Gamma \subseteq \mathcal{L}^d_{\Diamond\mathbf{A}}$ of declarative formulas is a complete declarative theory (CDT) if Γ is consistent, closed under deduction of declaratives, and complete with respect to declaratives (i.e. for any declarative α, either $\alpha \in \Gamma$ or $\neg\alpha \in \Gamma$).

Let Γ be a CDT. Define the preorder $\leq_\Gamma$ (global Γ-entailment) and the equivalence relation $\sim_\Gamma$ (global Γ-equivalence) on the set of declarative formulas $\mathcal{L}^d_{\Diamond\mathbf{A}}$ as follows:

$$\begin{array}{lll} \alpha \leq_\Gamma \beta & \text{iff} & \Gamma \vdash_{\mathbf{G}} \mathbf{A}(\alpha \to \beta); \\ \alpha \sim_\Gamma \beta & \text{iff} & \alpha \leq_\Gamma \beta \text{ and } \beta \leq_\Gamma \alpha \text{ (equivalent to } \Gamma \vdash_{\mathbf{G}} \mathbf{A}(\alpha \leftrightarrow \beta)). \end{array}$$

Definition (Theory-filtered resolutions)

Let Γ be a CDT and φ any formula. The Γ-filtered set of resolutions of φ is the set

$$\mathcal{R}_\Gamma(\varphi) := \{\alpha \in \mathcal{R}(\varphi) \mid \text{ for all } \beta \in \mathcal{R}(\varphi) : \alpha \leq_\Gamma \beta \text{ implies } \alpha \sim_\Gamma \beta\}.$$

Syntactic characterization of $\Diamond$

Lemma

Let φ be a formula and let Γ be a CDT. Then

$$\Gamma \vdash_{\mathbf{G}} \Diamond\varphi \quad \textit{iff} \quad \Gamma \vdash_{\mathbf{G}} \bigwedge_{\alpha \in \mathcal{R}_{\Gamma}(\varphi)} \Diamond\alpha.$$

Canonical model

Definition (Canonical general model)

The canonical general model is the structure $\mathcal{M}^c = (W^c, S^c, R^c, V^c)$ such that

- W^c is the set of CDTs;
- for all $\Delta, \Delta' \in W^c$: $\Delta' \in S^c[\Delta]$ iff for all $\alpha \in \mathcal{L}^d_{\Diamond\mathbf{A}}$, if $\mathbf{A}\alpha \in \Delta$ then $\alpha \in \Delta'$;
- for all $\Delta, \Delta' \in W^c$: $\Delta' \in R^c[\Delta]$ iff for all $\alpha \in \mathcal{L}^d_{\Diamond\mathbf{A}}$, if $\alpha \in \Delta'$ then $\Diamond\alpha \in \Delta$;
- $V^c(p) = \{\Delta \in W^c \mid p \in \Delta\}$.

Truth lemma

Lemma (Truth lemma for declaratives)

For any declarative formula $\alpha \in \mathcal{L}^d_{\Diamond\mathbf{A}}$ *and any* $\Delta \in W^c$*:* $\mathcal{M}^c, \Delta \models_{\mathbf{G}} \alpha$ *if and only if* $\alpha \in \Delta$.

Proof of truth lemma

By induction on the structure of declarative formulas. Most interesting case: $\Diamond\varphi$.

- Let $\Delta \in W^c$.
- By IH it holds that for all $\alpha, \beta \in \mathcal{R}(\varphi)$: $\alpha \leq_{\mathcal{M}^c[\Delta]} \beta$ iff $\alpha \leq_\Delta \beta$.
- Then $\mathcal{R}_{\mathcal{M}^c[\Delta]}(\varphi) = \mathcal{R}_\Delta(\varphi)$.
- Then, using IH and standard canonical model properties: $\mathcal{M}^c, \Delta \models_{\mathbf{G}} \Diamond\alpha$ iff $\Delta \vdash_{\mathbf{G}} \Diamond\alpha$ for all $\alpha \in \mathcal{R}_{\mathcal{M}^c[\Delta]}(\varphi) = \mathcal{R}_\Delta(\varphi)$.
- Then:

$$
\begin{array}{llll}
\mathcal{M}^c, \Delta \models_{\mathbf{G}} \Diamond\varphi & \text{iff} & \mathcal{M}^c, \Delta \models_{\mathbf{G}} \bigwedge_{\alpha \in \mathcal{R}_{\mathcal{M}^c[\Delta]}(\varphi)} \Diamond\alpha & \text{(by result on previous slide)} \\
& \text{iff} & \Delta \vdash_{\mathbf{G}} \bigwedge_{\alpha \in \mathcal{R}_\Delta(\varphi)} \Diamond\alpha & \text{(by above)} \\
& \text{iff} & \Delta \vdash_{\mathbf{G}} \Diamond\varphi & \text{(by result on previous slide)} \\
& \text{iff} & \Diamond\varphi \in \Delta. &
\end{array}
$$

Soundness and completeness

Theorem (Soundness and completeness, general semantics)

For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{\Diamond\mathbf{A}}$*:* $\Phi \models_{\mathbf{G}} \psi$ *if and only if* $\Phi \vdash_{\mathbf{G}} \psi$*.*

Axiomatizing logical consequence in standard semantics

An axiomatization of logical consequence in standard semantics is obtained by replacing the axiom schema (where α is required to be a declarative formula)

$$\mathbf{A}\alpha \to \alpha \tag{2.3}$$

by the schema

$$\mathbf{A}\varphi \to \varphi \tag{2.3*}$$

where φ can now be any formula.

Denote derivability in the resulting axiom system by $\vdash_{\mathsf{S}}$, and define $\Phi \vdash_{\mathsf{S}} \psi$ in the same way as before.

Axiomatizing logical consequence in standard semantics

Lemma

Let $\mathcal{M} = (W, S, R, V)$ be a general model, let $\emptyset \neq X \subseteq W$, and assume that $\mathcal{M}, X \models_{\mathbf{G}} \mathbf{A}\chi \to \chi$ for all $\chi \in \mathcal{L}_{\Diamond\mathbf{A}}$.

Let $\mathcal{M}' = (W', R', V')$ be the standard model such that $W' = \bigcup_{w \in X} S[w]$ and R' and V' are the restrictions of R and V to W'.

Then for all $Y \subseteq W'$, for all $\varphi \in \mathcal{L}_{\Diamond\mathbf{A}}$: $\mathcal{M}', Y \models \varphi$ iff $\mathcal{M}, Y \models_{\mathbf{G}} \varphi$.

Proposition

For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{\Diamond\mathbf{A}}$: $\Phi \models \psi$ implies $\Phi \cup \{\mathbf{A}\chi \to \chi \mid \chi \in \mathcal{L}_{\Diamond\mathbf{A}}\} \models_{\mathbf{G}} \psi$.

Axiomatizing logical consequence in standard semantics

Theorem (Soundness and completeness, standard semantics)

For any $\Phi \cup \{\psi\} \subseteq \mathcal{L}_{\Diamond\mathbf{A}}$: $\Phi \models \psi$ if and only if $\Phi \vdash_{\mathbf{S}} \psi$.

Proof of completeness:

$$
\begin{array}{llll}
\Phi \models \psi & \text{implies} & \Phi \cup \{\mathbf{A}\chi \to \chi \mid \chi \in \mathcal{L}_{\Diamond\mathbf{A}}\} \models_{\mathbf{G}} \psi & \text{(previous slide)} \\
& \text{implies} & \Phi \cup \{\mathbf{A}\chi \to \chi \mid \chi \in \mathcal{L}_{\Diamond\mathbf{A}}\} \vdash_{\mathbf{G}} \psi & \text{(completeness for general semantics)} \\
& \text{implies} & \Phi \cup \{\mathbf{A}\chi \to \chi \mid \chi \in \mathcal{L}_{\Diamond\mathbf{A}}\} \vdash_{\mathbf{S}} \psi & (\vdash_{\mathbf{S}} \text{ conservative extension of } \vdash_{\mathbf{G}}) \\
& \text{implies} & \Phi \vdash_{\mathbf{S}} \psi & (\vdash_{\mathbf{S}} \mathbf{A}\chi \to \chi \text{ for each } \chi \in \mathcal{L}_{\Diamond\mathbf{A}})
\end{array}
$$

Booth's minimal cover modality

The minimal cover modality

Another generalization of the existential Kripke modality is suggested by [Booth 2022].

Booth's generalization is designed to capture independence inferences:

> *If you may take an apple or a pear, you may take an apple without taking a pear, and you may take a pear without taking an apple.*

Booth's interpretation of $\Diamond$ is intended to capture the following property (assuming that the truth sets of p and q are not included in each other):

$$\Diamond(p \vee\!\!\!\vee q) \to \Diamond(p \wedge \neg q) \wedge \Diamond(q \wedge \neg p).$$

The minimal cover modality

Under Booth's interpretation: for $\Diamond\varphi$ to be true, each alternative for φ must be possible independently from the other alternatives for φ.

Definition (Minimal cover)

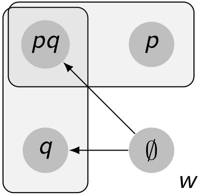
Let W be a set, let $C \subseteq \mathcal{P}(W)$ be a set of subsets of W, and let $X \subseteq W$. Then C is a minimal cover of X iff

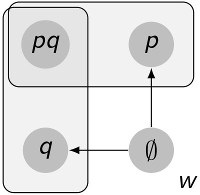
1. C is a cover of X, i.e. $X \subseteq \bigcup C$, and
2. no proper subset $C' \subset C$ is a cover of X.

Replace the clause previously used to interpret $\Diamond$ by the following:

$$\mathcal{M}, X \models_{\textbf{MC}} \Diamond\varphi \quad \text{iff} \quad \text{for all } w \in X \text{, there is a non-empty } Y \subseteq R[w] \text{ such that } \textbf{alt}_{\mathcal{M}}(\varphi) \text{ is a minimal cover of } Y.$$

Example





$\{|q|_{\mathcal{M}}\} \subset \mathbf{alt}_{\mathcal{M}}(p \vee\!\!\vee q)$ is a cover of $R[w]$.

$\mathbf{alt}_{\mathcal{M}}(p \vee\!\!\vee q)$ is not a minimal cover of $R[w]$.

$\mathcal{M}, w \not\models_{\mathbf{MC}} \Diamond(p \vee\!\!\vee q)$

No $C \subset \mathbf{alt}_{\mathcal{M}}(p \vee\!\!\vee q)$ is a cover of $R[w]$.

$\mathbf{alt}_{\mathcal{M}}(p \vee\!\!\vee q)$ is a minimal cover of $R[w]$.

$\mathcal{M}, w \models_{\mathbf{MC}} \Diamond(p \vee\!\!\vee q)$

Some validities

Where α is a declarative:

$$\models_{\mathbf{MC}} \Diamond(\alpha \mathbin{\mathbb{V}} \varphi) \land \neg \mathbf{A}(\alpha \to \varphi) \to \Diamond(\alpha \land \neg\varphi)$$

For each $k \in \mathbb{N}$ with $k \geq 2$, where α_i and α_j for $i, j \in \mathbb{N}$ are declaratives:

$$\models_{\mathbf{MC}} \left(\bigwedge_{1 \leq i \leq k} \Diamond \left(\alpha_i \land \bigwedge_{1 \leq j \leq k, j \neq i} \neg \alpha_j \right) \right) \to \Diamond \left(\bigvee\!\!\!\!\bigvee_{1 \leq i \leq k} \alpha_i \right).$$

Axiomatizing the minimal cover modality

1. Axioms for prop. inq. logic.
2. Axioms for $\mathbf{A}$, where $\alpha \in \mathcal{L}^{d}_{\Diamond\mathbf{A}}$ is declarative:
   2.1 $\mathbf{A}(\varphi \mathbin{\vee\!\!\!\vee} \psi) \to \mathbf{A}\varphi \vee \mathbf{A}\psi$
   2.2 $\mathbf{A}(\varphi \to \psi) \to (\mathbf{A}\varphi \to \mathbf{A}\psi)$
   2.3 $\mathbf{A}\varphi \to \varphi$
   2.4 $\mathbf{A}\varphi \to \mathbf{A}\mathbf{A}\varphi$
   2.5 $\neg\mathbf{A}\varphi \to \mathbf{A}\neg\mathbf{A}\varphi$
3. Axioms for $\Diamond$, where α, β, α_i and α_j for $i, j \in \mathbb{N}$ are declaratives:
   3.1 $\Diamond(\alpha \vee \beta) \leftrightarrow \Diamond\alpha \vee \Diamond\beta$
   3.2 $\Diamond\alpha \to \neg\mathbf{A}\neg\alpha$
   3.3 $\mathbf{A}(\varphi \leftrightarrow \psi) \to (\Diamond\varphi \leftrightarrow \Diamond\psi)$
   3.4 $\Diamond(\alpha \mathbin{\vee\!\!\!\vee} \varphi) \wedge \neg\mathbf{A}(\alpha \to \varphi) \to \Diamond(\alpha \wedge \neg\varphi)$
   3.5 For each $k \in \mathbb{N}$ with $k \geq 2$:

$$\left(\bigwedge_{1 \leq i \leq k} \Diamond \left(\alpha_i \wedge \bigwedge_{1 \leq j \leq k, j \neq i} \neg\alpha_j \right) \right) \to \Diamond \left(\mathop{\vee\!\!\!\vee}_{1 \leq i \leq k} \alpha_i \right)$$

Rules of inference:

- MP: from φ and $\varphi \to \psi$, infer ψ
- Nec for $\mathbf{A}$: if φ is a theorem, infer $\mathbf{A}\varphi$

Denote derivability in this system by $\vdash_{\mathbf{MC}}$, and define $\Phi \vdash_{\mathbf{MC}} \psi$ in the same way as before.

Proving completeness

The completeness proof follows the same structure as before:

- First define a general semantics for the logic, with a notion $\models_{\mathsf{GMC}}$ of logical consequence over general models.
- Take the axiom system on the previous slide and replace the schema $\mathbf{A}\varphi \to \varphi$ by the schema $\mathbf{A}\alpha \to \alpha$, where α is required to be a declarative, thus obtaining a notion $\vdash_{\mathsf{GMC}}$ of derivability.
- Prove completeness for general logical consequence, analogously to how it was proven earlier ($\Phi \models_{\mathsf{GMC}} \psi$ implies $\Phi \vdash_{\mathsf{GMC}} \psi$).
- Finally, general logical consequence can be shown to imply standard logical consequence in the same way as before, thus completeness for standard logical consequence is obtained.

Thank you!

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